<table>
<thead>
<tr>
<th>Date</th>
<th>To whom paid and for what object</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>1864, June 30</td>
<td>J. B. Grooms: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
<td>$2.50</td>
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<tr>
<td></td>
<td>W. Miller: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>T. W. Palmer: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>P. B. Finch: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>W. Mackin: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>J. L. Mitchell: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>W. M. Atherton: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>Z. B. Vance: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>The Asbury Times: For the Times furnished during the first session of the Forty-eighth Congress to Hon. E. H. Vance, December 4, 1864, to June 30, 1864.</td>
<td>$2.50</td>
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<td>H. H. Anthony: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>E. Hall: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>S. L. Colman: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>C. H. Van Wyck: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>E. G. Lapham: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>W. J. Sewall: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>G. P. Edmonds: For computation of allowance for stationery and newspapers for the fiscal year ending June 30, 1864.</td>
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<td>The Chicago Journal: For the Chicago Journal furnished during the first session of the Forty-eighth Congress to Hon. John A. Logan, December 4, 1864, to June 30, 1864.</td>
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<td>The Philadelphia Press Company: For the Daily Press furnished during the first session of the Forty-eighth Congress to the following named members of the United States Senate: Hon. W. J. Aldrich, March 31 to June 30, 1864.</td>
<td>$2.50</td>
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On Double Algebra. By Professor Cayley:

[Read April 2nd, 1884.]

1. I consider the Double Algebra formed with the extraordinary symbols, or "extraordinaries" $x, y$, which are such that

\[
\begin{align*}
    x^2 &= ax + by, \\
    y^2 &= cx + dy, \\
    xy &= ex + fy, \\
    yx &= gx + hy,
\end{align*}
\]

or, as these equations may also be written,

\[
\begin{vmatrix}
    x & y \\
    ax + by & cx + dy \\
    ex + fy & gx + hy
\end{vmatrix}
\]

where $a, b, c, d, e, f, g, h$ are ordinary symbols, or any coefficients; all coefficients being commutative and associative inter se, and with the extraordinaries $x, y$.

The system depends in the first instance on the eight parameters $a, b, c, d, e, f, g, h$; but we may, instead of the extraordinaries $x, y$, consider the new extraordinaries connected therewith by the linear relations $t = ax + by, s = y = ex + fy$, where the coefficients $a, b, c, d$ may be determined as to establish between the eight parameters any four relations at pleasure (or what is the same thing, $a, b, c, d$ are what I call "apercus, constantes"); and the number of parameters is thus properly $8 - 4 = 4$.
2. The extraordinary here considered are not in general associative; differing herein from the imaginaries of Peirce's Memoir, "Linear Associative Algebra" (1870), reprinted in the American Mathematical Journal, t. 8 (1881), pp. 97—227, which, as appears by the title, refers only to associative imaginaries. I recall some definitions and results.

The symbol $a$ is said to be idempotent if $a^2 = a$, nilpotent if $a^2 = 0$; and the systems of associative symbols are expressed as much as may be by means of such idempotent and nilpotent symbols: thus the linear systems are $(a_x) x^2 = x$, $(b_y) y^2 = 0$. A double system composed of independent symbols, that is, symbols $x, y$ each belonging to its own linear system, and moreover such that $xy = yx = 0$, is said to be "mixed"; thus the mixed double systems are

$$
\begin{array}{c|c}
  x & y \\
  \hline
  z & x \\
  y & z \\
\end{array}
\quad
\begin{array}{c|c}
  x & y \\
  \hline
  z & x \\
  y & z \\
\end{array}
\quad
\begin{array}{c|c}
  x & y \\
  \hline
  z & x \\
  y & z \\
\end{array}
$$

But these Peirce excludes from consideration, attending only to the pure systems, which he finds to be

$$(a_x) x = y, \quad (b_y) x = y, \quad (c_z) x = y, \quad (d_y) x = y$$

To these, however, should be added the system

$$\begin{array}{c|c}
  x & y \\
  \hline
  z & 0 \\
  y & 0 \\
\end{array}$$

3. In the general theory, where the symbols are not in the first instance taken to be associative, we may of course establish between the coefficients such relations as will make the symbols associative, and the question presents itself to show how in this case the system reduces itself to one of Peirce's systems. This I considered in my note "On Associative Imaginaries," Johns Hopkins University Circular, No. 15 (1882), p. 211; there obtained as the general form of the commutative and associative system

$$x^2 = ax + by, \quad xy = yx = ca + dy, \quad y^2 = b a + \frac{a^2 + bc - ad}{b} y.$$
7. In the commutative case, \( c = e = e \), and \( d = f \), we have

\[ \phi_2 = \Phi_2 = (ch - df) x^2 + (-ah - be + af + db)y^2 + (ad - bc) g^2; \]

here \( \xi_2 = 0 \), \( \eta_2 = 0 \), and the values may be taken to be \( (\xi, \eta) = (\xi, \eta) \),

\[ (\xi, \eta) = (\xi, \eta), \]

that is, \( \Phi = \Phi = \eta \), as above.

The value of \( \Omega \) is

\[ \Omega = g^2 + (b - 2c) x y^2 + (a - 2d) x^2 y^2 + b^2. \]

8. In the commutative and associative case, taking \( \xi, \eta, \xi \) to be the three idempotent symbols, \( \xi', \eta', \xi' \) to be, we have

\[ \xi' (\eta - \xi' \eta) = \eta' (\xi - \xi' \xi) = 0; \]

and in this manner we have the six equations

\[ \xi' (\eta - \xi' \eta) = 0, \quad \eta (\xi - \xi' \xi) = 0; \quad \eta (\xi - \xi' \xi) = 0; \quad (\xi - \xi' \xi) = 0. \]

viz., regarding the right-hand factor as being in each case expressed as a linear function of \( x, y \), we have apparently six products of two linear factors, each \( = 0 \); there is only one such product \( \Phi = 0 \), hence, disregarding coefficients, each of the six products must be \( = \Phi \), or it must be identically \( = 0 \), viz., this will be the case if the second factor \( by = 0 \).

We hence conclude that two of the symbols \( \xi, \eta, \xi \), suppose \( \xi \) and \( \eta \) must be factors of \( \Phi \), viz., \( \Phi \) must be \( \xi \eta \). We have \( \Omega = \xi^2 \); consequently \( \Omega = \xi^2 \eta^2 \), that is, two of the three linear factors of \( \Omega \) are the symbols \( \xi, \eta \), which are such that \( \xi \eta = \xi^2 \eta^2 = 0 \). To complete the theory, observe that \( \xi \) must be a linear function of \( \xi, \eta = \xi^2 + by \) suppose \( a, b \) coefficients, neither of them \( = 0 \); we hence have

\[ \xi^2 = \xi^2 + by^2 = \xi^2 + b \eta = \xi^2 + by; \]

that is, \( (\xi - a) \xi + (b - d) \eta = 0 \); whence \( a = 0, d = 1 \), and therefore \( \xi = \xi + \eta \); hence also \( \xi \xi = \xi \) and \( \xi^2 = \eta \xi = \eta \xi^2 = \xi \); the six products consequently are \( \xi \eta, \xi^2, \xi^2 \eta, \xi^2 \eta^2, \xi \eta \), each \( = \Phi \) identically \( = 0 \).

9. In verification of the theorem that for the commutative and associative system the cubic function \( \Omega \) contains the quadratic function \( \phi \) as a factor, we may write, as above,

\[ y = \frac{cd}{b}, \quad k = \frac{a^2 + bc - ad}{b}; \]

values which give

\[ b \Phi = \Phi (b^2 - acd) x^2 + (-a^2 - ab + ac - be + db) x y + (ac + b) y^2; \]

\[ = (cd - b) \{ a^2 + (cd - b) x y - b y^2 \}; \]

\[ b \Omega = a^2 \xi + (a - b) a^2 + b^2 \xi + (ab - bd) x y + b y^2 = (a^2 + a^2) x y - b y^2; \]

\[ = (a - b) \{ a^2 + (cd - b) x y - b y^2 \}; \]
which gives the theorem in question. And observe further that
\[
(dx - by)^2 = \frac{c^2}{y^2} \left( x^2 + 4y^2 - 4x^2 - 4y^2 + 4y^2 \right),
\]
\[
= \frac{c^2}{y^2} \left( ad - bc \right) y^2.
\]
That is, disregarding coefficients, the two idempotent symbols \( \xi, \eta \) are the linear factors of \( x^2 + (d-a)xy - by^2 \), and the third idempotent symbol \( \zeta \) is \( dx - by \).

10. Introducing coefficients in order to make the symbols \( \xi, \eta, \zeta \) idempotent, and writing accordingly:
\[
\xi = \frac{1}{K} \left( \frac{e}{\sqrt{V}} \right) \left( x + \frac{1}{2} (d-a) + \sqrt{V} y \right), \quad \eta = \frac{1}{L} \left( \frac{e}{\sqrt{V}} \right) \left( x + \frac{1}{2} (d-a) - \sqrt{V} y \right),
\]
\[
\zeta = \frac{1}{L} \left( dx - by \right),
\]
we have to verify that it is possible to determine \( K, L, P \) so that \( \xi^2 = \xi, \eta^2 = \eta, \xi^2 = \zeta, \zeta = \xi + \eta \). The last equation gives
\[
\frac{dP}{P} = \frac{e}{K} + \frac{\xi}{L},
\]
\[
= \frac{d^2}{P} \left( \frac{e}{\sqrt{V}} \right) \left( x + \frac{1}{2} (d-a) + \sqrt{V} y \right) + \frac{\xi}{L},
\]
and we then have
\[
\frac{d}{P} \left( \frac{d}{2} \right) \frac{d(a-a) + 2bc + \sqrt{V}}{d^2} \frac{d}{L},
\]
and we can from the equation \( \xi^2 = \xi \) find \( P \); viz., comparing the coefficients of \( y \), we have
\[
\frac{d}{P} = \frac{1}{L} \left( d(a-a) + 2bc + \sqrt{V} \right) = \frac{d}{L} \left( ad - bc \right),
\]
or the values of \( K \) and \( L \) are
\[
\frac{d}{P} \left( \frac{d}{2} \right) \frac{d(a-a) + 2bc + \sqrt{V}}{d^2} \frac{d}{L},
\]
which should agree with the values of \( K \) and \( L \) found from the equations \( \xi^2 = \xi, \eta^2 = \eta \), respectively. Comparing the coefficients of \( y \),
But the equations are obtained in a more simple form by taking \( x \) for the ideon and \( y \) for the nil; viz., we then have \( x^2 = x, \ y^2 = cx + dy, \ y = cx + dy, \ y^2 = 0 \); we must then have \( z = -(e + c)x + (1 - d - f)y \) for an ideon; this gives \( z^2 = -(e + c)(d + f)z \), and we have the negative conditions \( e + c \neq 0, d + f \neq 0 \) or 1.

(3) 1 ideon and 2 nils. This may be deduced from (2) by writing therein \( d + f = 0 \); for then \( z = -(e + c)x + y \) is a nil. The equations are \( x^2 = x, \ y^2 = cx + dy, \ y^2 = 0 \); and if, instead of \( z \), we introduce therein \( z \) by the equation \( z = -(e + c)x + y \), the equations become \( y^2 = 0, \ y^2 = [c - d(e + c)]y^2 + c, \ y = 0, \ y = 0 \), with the negative condition \( e + c \neq 0 \).

But it is more simple to take \( x, y \) as the nils: the equations then are \( x^2 = 0, \ x^2 = cx + dy, \ y^2 = 0 \). We must have \( z = (e + c)x + (d + f)y \), an ideon: this gives \( z^2 = (e + c)(d + f)z \), and we have the negative conditions \( e + c \neq 0, d + f \neq 0 \).

(4) We cannot have three nils. For in (3) to make \( z \) a nil we must have \( e + c = 0 \) or \( d + f = 0 \), and in the two cases \( z = (e + c)x + (d + f)y \) becomes \( = x \) and \( = y \); so that \( x \) or \( y \) is a twofold nil.

Or, what comes to the same thing, we have

\[ \Omega = -(e + c) x y - (d + f) x y^2, \]

and \( \Omega \) has a twofold factor if \( e + c = 0 \) or \( d + f = 0 \).

(4) A twofold ideon and a onefold ideon. Taking \( x \) for the twofold ideon and \( y \) for the onefold ideon, \( \Omega \) must reduce itself to \( (1 - c \cdot e - d \cdot f) x y \), viz., we must have \( d + f = 1 \), or \( e + f = 1 \). The equations are \( x^2 = x, \ y^2 = cx + dy, \ y^2 = 0 \); and we have the negative condition \( e + c \neq 1, d + f \neq 1 \); otherwise \( \Omega \) would vanish identically.

(5) A twofold ideon and a onefold nil. Taking \( x \) for the twofold ideon and \( y \) for the onefold nil, then the equations are \( x^2 = x, \ y^2 = cx + dy, \ y^2 = 0 \); and we have the negative condition \( e + c \neq 0 \).

(6) A twofold nil and a onefold ideon. Taking these to be \( x \) and \( y \), then \( d + f = 0 \), and the equations are \( x^2 = 0, \ y = cx + dy, \ y^2 = 0 \), and we have the negative condition \( e + c \neq 0 \).

(7) A twofold nil and a onefold nil. Taking these to be \( x \) and \( y \), we have \( d + f = 0 \), and the equations are \( x^2 = 0, \ y = cx + dy, \ y^2 = 0 \); with the negative condition \( e + c \neq 0 \).

(8) A threefold ideon. Taking this to be \( x \), then \( \Omega \) must reduce itself to \( g x^2 \), viz., we must have \( h = c + e, \ 1 = d + f \); and the equations are \( x^2 = x, \ y = cx + dy, \ y^2 = cx + (1 - d - f)y \); we have the negative condition \( g \neq 0 \), for otherwise \( \Omega \) would vanish identically.

(9) A threefold nil. Taking this to be \( z \), then we must have \( h = c + e, \ 0 = d + f \); the equations are \( x^2 = 0, \ y = cx + dy, \ y^2 = cx + dy, \ y^2 = 0 \); and there is again the negative condition \( g \neq 0 \).

(10) \( \Omega \) = 0 identically: infinity of ideons, 1 nil. \( \Omega \) will vanish identically if \( g = 0, \ h = c + e, \ a = d + f, \ b = 0 \). If there is 1 ideon, there will be an infinity of ideons, and 1 nil. For, assume an ideon \( x \), \( x^2 = x \); and, if possible, let there be no other ideon; then there will be a nil \( y, \ y^2 = 0 \). We have \( c + e = 0, \ d + f = 1 \), and the equations are \( x^2 = x, \ xy = cx + dy, \ y^2 = 0 \); whence \( x y + y^2 = y \).

Taking \( \alpha, \beta \) arbitrary coefficients, we have

\( (a x + \beta y)^2 = a^2 x + \beta^2 y = (a + \beta)(a x + \beta y) \);

hence \( a + \beta y \) is an ideon, except in the case \( a = 0 \), when it is the original nil \( y \).

If besides the ideon \( x \) we have an ideon \( y \), then the conditions are \( c + e = 1, \ d + f = 1 \); the equations are

\( x^2 = x, \ xy = cx + dy, \ y^2 = 0 \); and we have the negative condition \( e + c \neq 1 \), for otherwise \( \Omega \) would vanish identically.

(11) \( \Omega = 0 \) identically: an infinity of nils. Taking the two nils \( x \) and \( y \), the conditions are \( e + c = 0, \ d + f = 0 \); the equations are \( x^2 = 0, \ xy = cx + dy, \ y^2 = 0 \); whence \( x y + y^2 = 0 \).

Considering the arbitrary combination \( a x + \beta y \), we have

\( (a x + \beta y)^2 = a x^2 + \beta x y + \beta y^2 = (a + \beta)(a x + \beta y) \).

This is an ideon, except in the case \( a + \beta = 0 \), when it is a nil; or, in the case \( a = 0 \), we have the single nil \( y \). We have thus again an infinity of ideons, 1 nil.

12. The different cases may be grouped together as follows:

A. 2 ideons, (1), (2), (4), (10).

Equations \( x^2 = x, \ xy = cx + dy, \ y^2 = 0 \).

B. 1 ideon and 1 nil, (3), (5), (6), (10).

Equations \( x^2 = x, \ xy = cx + dy, \ y^2 = 0 \).

C. 2 nils, (5), (7), (11).

Equations \( x^2 = 0, \ xy = cx + dy, \ y^2 = 0 \).

D. Threefold ideon, (8).

Equations \( x^2 = x, \ xy = cx + dy, \ y^2 = cx - (1 - d)y \).
E. Thricefold nil, (9).
Equations \( x^2 = 0, \ xy = cx + dy, \ yz = cx - dy, \)
\( y^2 = gx + (c + e) y. \)

The several cases of \( A, B, C \) respectively are distinguished by negative conditions which need not be here repeated.

12. I consider, as in my Note before referred to, the conditions in order that the system may be associative. We have the 8 products,
\( x^2, x^2, xy, y^2, y^2, yz, z^2, \) giving rise to equations \( x^2 = x^2, x, \)
\( xy = x^2, ..., y^2 = y^2, \) which, on putting therein for \( x^2, xy, yz, \)
their values, must be satisfied identically. We thus obtain in the first instance 16 relations, but some of these are repeated, and we have actually only 12 relations; viz., the relations are

\[
\begin{align*}
\text{(twice)} & \quad b \ (x-e) = 0, \\
\text{(twice)} & \quad g \ (f-d) = 0, \\
\text{(twice)} & \quad g \ (j-d) = 0, \\
\text{(twice)} & \quad b g = c d = 0, \\
\text{(twice)} & \quad b g = c^2 = 0, \\
& \quad c (c-b) + y (d-a) = 0, \\
& \quad d (d-a) + b (c-h) = 0, \\
& \quad c (c-h) + g (f-a) = 0, \\
& \quad b (c-h) + f (f-a) = 0, \\
& \quad c (c-e) = g f + e c = 0, \\
& \quad b (f-j) = c f + d c = 0.
\end{align*}
\]

14. From the first four equations, it appears that either \( b = 0, \)
\( g = 0, \) or else \( c = e \) and \( d = f. \) I attend first to the latter case, viz., we have here the commutative system

\[
x^2 = x a + b y, \ xy = y z = x c + d y, \ y^2 = g y + b y.
\]

In order that this may be associative, we must still have the relations
\[
bg = cd = 0, \\
c (c-h) + g (d-a) = 0, \\
d (d-a) + b (c-h) = 0,
\]
or, as they may be written

| b, -c, d-a | = 0.

15. First, if the form be \( A, B, C \); there will be the two idem-ordinary symbols \( e \) and \( y, \) that is, we may assume \( b = 0, \ g = 0; \) and the associative conditions then become \( cd = 0, \ (e-h) = 0, \) \( d (d-a) = 0, \) viz., for the forms \( A, B, C, \)

\[
\begin{align*}
A. & \quad x^2 = x, \ y^2 = y, \\
B. & \quad x^2 = x, \ y^2 = y, \\
C. & \quad x^2 = 0, \ y^2 = 0,
\end{align*}
\]

these are \( cd = 0, \ (e-1) = 0, \) \( d (d-a) = 0; \) \( c = 0 \) or \( 1, \ d = 0 \) or \( 1, \)

\[
\begin{align*}
\text{or} & \quad cd = 0, \\
& \quad c = 0, \ d (d-a) = 0; \ c = 0, \ d = 0 \ or \ 1, \\
& \quad cd = 0, \\
& \quad c = 0, \ d = 0; \ c = 0, \ d = 0.
\end{align*}
\]

But for the form \( A, \) if \( e = 0, \ d = 1, \) that is, \( xy = y z = y, \) then, writing \( z = x-y, \) we have \( x^2 = x, \) \( y = y, \) \( y^2 = y. \) And similarly, if \( e = 1, \)
\( d = 0, \) that is, \( x = y = x, \) then, writing \( z = x-y, \) we have \( x^2 = x, \)
\( x = y = x, \) \( y = y, \) \( y^2 = y. \) That is, each of these is reduced to the first case \( e = 0, \ d = 0; \) that is, \( x^2 = x, \)
\( y = y = x, y^2 = 0, y^2 = y. \)

For the form \( B, \) if \( e = 0, \ d = 1, \) then the system is \( x^2 = x, \)
\( y = x z = y, y^2 = 0; \) and this cannot be reduced to the first case \( x^2 = x, \) \( y = y = 0, \) \( y^2 = 0. \)

For the form \( C, \) there is only one case, as above.

For the form \( E, \) we have \( e = 0, \ b = 0, \ c = d = 0, \) in order that the system may be commutative, \( h = 2 e, \) viz., the equations must be \( x^2 = 0, \) \( y = y = 0, \) \( y^2 = g z + 2 y. \) The associative conditions then give \( c = 0; \) or, the system is \( x^2 = 0, \) \( y = y = 0, \) \( y^2 = y. \)

Writing \( x, y \) instead of \( x, y; \) and for convenience interchanging \( x, y, \)

the equations are \( x^2 = y, \) \( y = y = 0, \) \( y^2 = 0. \)

16. The commutative associative system is thus seen to be reducible.
Professor Cayley on Double Algebra.  [April 3.

as follows:—
A. system is \( x^2 = x, \ xy = xy = 0, \ y^2 = y \), first mixed system see No. 2.
\( x^2 = x, \ xy = xy = y, \ y^2 = 0 \), Peirce's system (\( a \)).

B. \( x^2 = x, \ xy = xy = 0, \ y^2 = 0 \), second mixed system.
C. \( x^2 = 0, \ xy = xy = 0, \ y^2 = 0 \), third mixed system.
D. \( x^2 = y, \ xy = xy = 0, \ y^2 = y, \) Peirce's system (\( c \)).

I said, at the end of my Note before referred to, that it had been pointed out to me that my system (the commutative associative system), in the general case \( ad - bc \neq 0 \), is expressible as a mixture of two algebras of the form \( (a) \), see p. 120; whereas, if \( ad - bc = 0 \), it is reducible to the form \( (c) \), see p. 122. The accurate conclusion is as above, that the commutative associative system is either a mixed system of one of the three forms, or else a system \( (a) \) or \( (c) \).

17. Considering next the non-commutative associative systems, we have here, \( a, b, c, d, e, f, g, h \); and the relations which remain to be satisfied then are
\[ cd = 0, \ ef = 0, \ (e - h) = 0, \ d(d - a) = 0, \ e(f - d) = 0, \ f(f - a) = 0, \ h(f - d) = 0, \]
\[ a(c - e) - e^2 + de = 0, \ b(f - d) - g^2 + de = 0. \]
The first equation gives \( cd = 0 \), that is, \( c = 0 \) or \( d = 0 \); but we may restrict exclusively to the case \( c = 0 \), for the case \( d = 0 \) may be deduced from this by the interchange of \( c, y \). We have then of \( = 0 \).

It will be convenient to separate the cases
I. \( c = 0, e = 0, f = 0 \), giving \( d - a = 0, \ db = 0, \)
II. \( c = 0, e = 0, f \neq 0 \),
\[ d(d - a) = 0, \ f - a = 0, \ h(f - d) = 0, \]
III. \( c = 0, e \neq 0, f = 0 \),
\[ d(d - a) = 0, \ e - h = 0, \ (d - a) = 0, \]
\[ d(f - d) = 0, \]
that is, \( d - a = 0, \ e - h = 0. \)

18. We have thus five cases
I. (a). \( d = 0 ; \ x^2 = x, xy = 0, \ y^2 = y \), commutative, and so included in what precedes.

II. (b). \( d = a, \ h = 0 ; \ x^2 = x, xy = 0, \ y^2 = 0 \); or, writing as we may do \( a = 1 \), this is \( x^2 = x, xy = y, \ y^2 = 0 \); which is Peirce's system (\( b \)).

II. (c). \( d = f = 0 ; x^2 = 0, xy = 0, \ y^2 = y = 0 \), commutative, and so included in what precedes.

II. (d). \( d = 0, f = a, \ h = 0 ; \ x^2 = x, xy = 0, \ y^2 = 0 \); or, writing as we may do \( a = 1 \), this is \( x^2 = 0, xy = y, \ y^2 = 0 \); which is the system (\( d \)).

19. It may be proper to show that the systems \( (b) \), \( x^2 = x, xy = y, \ y^2 = 0 \), and \( (d) \), \( x^2 = x, xy = y, \ y^2 = 0 \), or say
\[ a \ b \ c \ d \ e \ f \ g \ h \]
\( (b) \) 1 0 0 1 0 0 0 0
\( (d) \) 1 0 0 0 0 0 1 0 0, \[ c \ 27 \]
are really distinct from each other. Observe that they each belong to the case \( 10, \Omega = 0 \), an infinity of invariants and 1 nil; viz., in each of them writing \( z = x + \beta y \), \( \beta \) an arbitrary coefficient, we have \( x^2 = x + \beta(x + y), \ y^2 = x + \beta y, \ z = z \), and \( y \) is the only nil. And, this being so, we have in the first system \( xy = y, \ y^2 = 0 \); viz., the system is \( z^2 = z, \ y = y, \ y^2 = 0 \); retaining, when we write \( z \) for \( x \); its original form. And similarly, in the second system, \( xy = 0, \ y = y \); viz., the system is \( z^2 = z, \ y = 0, \ y^2 = 0 \); retaining, when we write \( z \) for \( x \), its original form. The two are thus distinct systems, in no wise transformable the one into the other.

On Electrical Oscillations and the effects produced by the motion of an Electrically Sphere. By J. J. Thomson, M.A., Fellow and Assistant Lecturer of Trinity College, Cambridge.

[Read April 3rd, 1881.]

In this paper two problems are discussed which, though physically different, are yet capable of solution by almost the same mathematical treatment.

The first problem treats of the vibrations which take place in the electrical distribution on the surface of a thin spherical shell when the