On the Ghosts in Rutherford's Diffraction-Spectra.

By C. S. PEIRCE.

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Let there be a periodical irregularity in the ruling of a diffraction plate, so that the side of the $r$th slit nearest a fixed line of reference parallel to the ruling shall be distant from that line by

$$r - 1 \frac{1}{2} a \cos \left( r \theta - 1 \frac{1}{2} \theta \right)$$

while the side of the same opening furthest from the line of reference is distant from it by

$$r + 1 \frac{1}{2} a \cos \left( r \theta + 1 \frac{1}{2} \theta \right).$$

This is supposing the opaque lines to have a constant breadth, $(1 - a) w$. Suppose the collimator and telescope of the spectrometer to be focused for parallel rays, and neglect the angular aperture of the slit. Let the angle of incidence be $i$, and the angle of emergence $j$. Write

$$r = \sin i - \sin j.$$

Then the ray which strikes the grating at a distance $x$ from the line of reference is longer than that which passes through the line of reference by $ax$. Consequently, the resultant oscillation from the $r$th slit will be

$$\int dr \sin 2 \frac{r}{\lambda} \pi \cos \left( r \theta + 1 \frac{1}{2} \theta \right)$$

where $t$ is the time, $V$ the velocity of light, and $\lambda$ the wave-length. (In this paper $\pi$ will be written for the ratio of the circumference to the diameter, $e$ for the natural base, and $i$ for the imaginary unit.) If we sum this for all integral values of $r$, we obtain an expression for the resultant oscillation from the whole grating.

Performing the integration relatively to $x$, indicating the summation relative to $r$, and using the abbreviations

$$\omega = 2 \frac{\pi}{\lambda}, \quad \omega = 2 \frac{\pi}{\lambda}, \quad \tau = 2 \frac{\pi}{\lambda},$$

we obtain the following expression for the resultant oscillation from the whole grating:

$$\sum_{m} \left[ \cos \left\{ \omega \sin \left( r \theta + 1 \frac{1}{2} \theta \right) \right\} - \cos \left\{ \omega \sin \left( r \theta - 1 \frac{1}{2} \theta \right) \right\} \right].$$

We now need a formula for developing sines and cosines of sines. For this purpose take $y = e^x$. Then we have

$$\cos (a \sin x) + \sin (a \sin x) = e^{iax} = e^{i \left( (-1)^{x} \right)}.$$

By the usual development of an exponential function, this is

$$e^{i \left( (-1)^{x} \right)} = \frac{e^{a}}{1} \left[ \frac{e^{a}}{1} \right] \left[ \frac{e^{a}}{1} \right] \left[ \frac{e^{a}}{1} \right] \left[ \frac{e^{a}}{1} \right] \left[ \frac{e^{a}}{1} \right] \left[ \frac{e^{a}}{1} \right] \left[ \frac{e^{a}}{1} \right] \left[ \frac{e^{a}}{1} \right] \left[ \frac{e^{a}}{1} \right],$$

and by the binomial theorem, this is

$$e^{i \left( (-1)^{x} \right)} = \frac{1}{2^{m}} \left( \frac{e^{a}}{1} \right) \left( \frac{e^{a}}{1} \right) \left( \frac{e^{a}}{1} \right) \left( \frac{e^{a}}{1} \right) \left( \frac{e^{a}}{1} \right) \left( \frac{e^{a}}{1} \right) \left( \frac{e^{a}}{1} \right) \left( \frac{e^{a}}{1} \right) \left( \frac{e^{a}}{1} \right) \left( \frac{e^{a}}{1} \right).$$

The $pg^{m}$ term is

$$-1^{x} \frac{e^{a}}{2^{m} \left( p + q \right) \left( m + q \right) \left( m + q \right) \left( m + q \right) \left( m + q \right) \left( m + q \right) \left( m + q \right) \left( m + q \right) \left( m + q \right) \left( m + q \right) \left( m + q \right).$$

In regard to the limits of the summation, $q$ may have any value from zero to positive infinity, and, for every value of $q$, $p$ may have any value from $q$ to positive infinity; hence, $m$ may have any value from $-q$ to positive infinity, and we have

$$\cos (a \sin x) + \sin (a \sin x) = \sum_{m} \left( (-1)^{x} \right) \left( \frac{e^{a}}{2^{m} \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right).$$

If $m$ has a positive value, $q$ may have any positive value; but if $m$ has a negative value, $q$ can only have any positive value greater than $-m$. Hence, we may take the terms for which $m$ is not zero in pairs, embracing in each pair a term for which $m$ has a positive value, $M$, and a term for which $m = -M$. The sum of two terms composing the pair is then,

$$\left( (-1)^{x} \frac{e^{a}}{2^{m} \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right) \left( q + m \right).$$

The sum of two terms composing the pair is then,
If $M$ is even, the value of this is

$$(-1)^M \alpha^M \frac{a^M}{2^{2r+1} 4Q(M+Q)} \cos M\alpha;$$

and if $M$ is odd, its value is

$$(-1)^M \alpha^M \frac{a^M}{2^{2r+1} 4Q(M+Q)} \sin M\alpha.$$

We have then

$$\cos(a \sin x) + \sin(a \sin x) = \sum_{r=0}^{\infty} \frac{(-1)^r a^{2r}}{2^{2r+1} 4\Gamma(1/2)} \alpha^{2r} (\cos x + \sin x)^r;$$

where

$$A_n = \sum_{r=0}^{n} \frac{(-1)^r}{4\Gamma(1/2+1)} \alpha^{2r}.$$

Performing the numerical calculations, we have

$$\cos(a \sin x) = 1 - \frac{1}{4} a^2 \frac{1}{12} a^4 + \frac{1}{204} a^6 + \frac{1}{171600} a^8 + \text{etc.} + \frac{1}{2} a^2 \left(1 - \frac{1}{12} a^2 + \frac{1}{384} a^4 - \frac{1}{23040} a^6 + \text{etc.} \right) \cos 2x + \frac{1}{192} a^4 \left(1 - \frac{1}{20} a^2 + \frac{1}{760} a^4 - \frac{1}{60480} a^6 + \text{etc.} \right) \cos 4x + \frac{1}{23040} a^6 \left(1 - \frac{1}{28} a^2 + \frac{1}{1728} a^4 - \text{etc.} \right) \cos 6x + \frac{1}{616960} a^8 \left(1 - \frac{1}{30} a^2 + \text{etc.} \right) \cos 8x + \frac{1}{18349560} a^{10} \left(1 - \text{etc.} \right) \cos 10x + \text{etc.}$$

$$\sin(a \sin x) = a - \frac{1}{8} a^3 - \frac{1}{12} a^4 \left(1 - \frac{1}{16} a^2 + \frac{2}{64} a^4 - \frac{2}{4096} a^6 + \frac{1}{16} a^8 \left(1 - \frac{1}{16} a^2 + \frac{1}{48} a^4 - \frac{1}{8} a^6 + \text{etc.} \right) \sin x - \frac{1}{64} a^6 \left(1 - \frac{1}{16} a^2 + \frac{2}{64} a^4 - \frac{2}{4096} a^6 + \frac{1}{616960} a^8 + \text{etc.} \right) \sin x + \frac{1}{192} a^8 \left(1 - \frac{1}{30} a^2 + \frac{1}{1728} a^4 - \text{etc.} \right) \sin 5x + \frac{1}{23040} a^{10} \left(1 - \frac{1}{40} a^2 + \text{etc.} \right) \sin 9x + \frac{1}{4} \sum_{n=1}^{\infty} a^{2n} \left(1 - \text{etc.} \right) \sin n\alpha + \text{etc.}$$

Making use of these series, the expression for the resultant oscillation from the gitter becomes

$$-w \sum_{r=0}^{\infty} A_r \Gamma(1/2+1) \alpha^{2r+1} \cos rM\alpha \cos \frac{1}{2} \alpha \sin \frac{1}{2} \alpha r \sin \frac{1}{2} \alpha + \sin \frac{1}{2} \alpha \sin \frac{1}{2} \alpha$$

$$+ \sin \frac{1}{2} \alpha \cos \frac{1}{2} \alpha \cos \frac{1}{2} \alpha$$

$$- w \sum_{r=1}^{\infty} (2r-1) A_r \Gamma(1/2+1) \alpha^{2r} \cos (r-1)M\alpha \sin \frac{1}{2} \alpha \sin \frac{1}{2} \alpha r \sin \frac{1}{2} \alpha + \sin \frac{1}{2} \alpha \sin \frac{1}{2} \alpha$$

$$+ \sin \frac{1}{2} \alpha \cos \frac{1}{2} \alpha \cos \frac{1}{2} \alpha$$

The summation relatively to $r$ may be effected by means of the formula,

$$\sum (a \pm b) = \frac{\sin \left(x + \frac{1}{2} \alpha \right) \cos \left(x + \frac{1}{2} \alpha \right)}{\cos \left(x + \frac{1}{2} \alpha \right)}$$

For a modern gitter, it would be quite as satisfactory to consider $a$ as an infinitesimal, and to use, in place of the above, an infinitesimal formula, which will be found in Hirsch's Integral Tables. Applying, however, the formula of finite integration, we have, as an integrated expression for the resultant oscillation from the whole gitter,

$$w A_0 \left(1 - \cos \alpha \sin \left(\cos r - \frac{1}{2} \alpha \right) \sin \frac{1}{2} \alpha - \cos \frac{1}{2} \alpha \sin \frac{1}{2} \alpha \right)$$

$$+ w \sum_{r=0}^{\infty} A_r \Gamma(1/2+1) \alpha^{2r+1} \left(\sin M\alpha \left(\cos \left(r - \frac{1}{2} \alpha \right) \sin \left(\cos \left(r - \frac{1}{2} \alpha \right) \sin \frac{1}{2} \alpha \sin \frac{1}{2} \alpha \right) \right) + \cos \left(\cos \left(r - \frac{1}{2} \alpha \right) \sin \left(\cos \left(r - \frac{1}{2} \alpha \right) \sin \frac{1}{2} \alpha \sin \frac{1}{2} \alpha \right) \right) \right)$$

$$+ w \sum_{r=0}^{\infty} A_r \Gamma(1/2+1) \alpha^{2r+1} \left(\sin \left(\cos \left(r - \frac{1}{2} \alpha \right) \sin \left(\cos \left(r - \frac{1}{2} \alpha \right) \sin \frac{1}{2} \alpha \sin \frac{1}{2} \alpha \right) \right) + \cos \left(\cos \left(r - \frac{1}{2} \alpha \right) \sin \left(\cos \left(r - \frac{1}{2} \alpha \right) \sin \frac{1}{2} \alpha \sin \frac{1}{2} \alpha \right) \right) \right)$$

This expression may be simplified by writing

$$x = \frac{1}{2} (\alpha + n \beta)$$

$$y = \frac{1}{2} (\alpha - n \beta)$$
the expression for the resultant oscillation from the whole gitter reduces to
\[ \sin \tau \cdot \frac{R}{\sin \frac{\pi}{2}} \cdot A_x \cdot \sin \frac{1}{2} R (\omega + m \delta) \cdot \sin \frac{1}{2} \frac{\pi}{2} (\omega + m \delta) \cdot \sin \frac{1}{2} (\omega \pm \omega \delta) \cdot \sin \frac{1}{2} \frac{\pi}{2} (\omega \pm \omega \delta), \]

where, in summing for negative values of \( m \), positive values are to be taken in the coefficients, and where terms arising from odd negative values of \( m \) in the parenthesis are to have the opposite sign, and where the term in \( m = 0 \) is to have only half the above value.

We have now to study the principal maxima of the amplitude of this oscillation, for varying \( \omega \). Taking each term of the series separately, we observe that one factor of it, namely,
\[ \sin \frac{1}{2} R (\omega + m \delta) \]
\[ \sin \frac{1}{2} \left( \omega + m \delta \right), \]

reaches a maximum when \( \omega + m \delta = 2N \pi \),

and this maximum value is \( R \). Now \( R \) is a number amounting to several thousand, while \( \omega \) is less than unity. Hence, the maximum of the whole term will be very nearly at the same place, and one of the maxima of the sum of all the terms will also be nearly in that place.

To ascertain the precise position of the maximum of any one term, put \( \omega = 2N \pi - m \delta + \lambda \omega \).

Then, neglecting the cube of \( \delta \omega \), in comparison with unity, we have
\[ \sin \frac{1}{2} R (\omega + m \delta) = \pm \sin \frac{1}{2} R \delta \omega = \pm \frac{1}{2} R \delta \omega \sin \frac{1}{48} \left( \delta \omega \right)^2 \]
\[ \sin \frac{1}{2} (\omega + m \delta) = \pm \sin \frac{1}{2} \delta \omega = \pm \frac{1}{2} \delta \omega \sin \frac{1}{48} \left( \delta \omega \right)^2 \]
\[ \sin \frac{1}{2} R (\omega + m \delta) = \pm \frac{1}{2} R \delta \omega \sin \frac{1}{48} \left( \delta \omega \right)^2 \]
\[ \sin \frac{1}{2} (\omega + m \delta) = \pm \frac{1}{2} \delta \omega \sin \frac{1}{48} \left( \delta \omega \right)^2 \]

As for \( \sin \frac{1}{2} \left( \omega + (\pm 1) m \delta \right) \), it may have any value whatever from \(-1\) to \(+1\), according to the magnitude of \( \omega \). But it is when it vanishes that the maximum is at the greatest value of \( \delta \omega \). Let us then suppose
\[ \sin \frac{1}{2} \left( \omega + (\pm 1) m \delta \right) = \pm \frac{1}{2} \delta \omega \sin \frac{1}{48} \left( \delta \omega \right)^2. \]
Finally, there is the factor $\omega^{-1}$. Dividing this by $(2N\pi - m\theta)^{-1}$, we have
\[
\left(\frac{\omega}{2N\pi - m\theta}\right)^{m-1} = 1 + \frac{(m-1)}{2} (2N\pi - m\theta)^{-1} \delta \omega + \frac{(m-1)(m-2)}{2} (2N\pi - m\theta)^{-2} \left(\frac{R - 1}{\omega}\right) \delta \omega^2.
\]

Finally, multiplying together the quantities thus obtained, we find as that factor of the $n$th term which contains $\delta \omega$
\[
\delta \omega + (m-1)(2N\pi - m\theta)^{-1} \delta \omega^2 + \frac{(m-1)(m-2)}{2} (2N\pi - m\theta)^{-2} \left(\frac{R - 1}{\omega}\right) \delta \omega^2 = 0.
\]

Differentiating, we find as the equation for determining the value of $\delta \omega$ at the maximum of the $n$th term
\[
1 + 2(m-1)(2N\pi - m\theta)^{-1} \delta \omega + \frac{3(m-1)(m-2)}{2} (2N\pi - m\theta)^{-2} \left(\frac{R - 1}{\omega}\right) \delta \omega^2 = 0.
\]
If we neglect $\frac{1}{\omega^2}$, the solution of this equation is
\[
\delta \omega = \frac{8\omega}{2N\pi - m\theta}.
\]

It will be seen that $\delta \omega$ is zero when $\omega = 0$, and that for the principal spectrum, for which $m = 0$, if $R = 1000$, $\delta \omega$ is altogether inappreciable, but if $R = 100$, $\delta \omega$ is about $\frac{1}{5000}$, for the first order, which displaces the spectrum by about $\frac{1}{50}$ part of the distance between the two $D$ lines.

We have now to consider how far the maxima of the sum of the series representing the oscillation may differ from those of the single terms. A term will have the most influence in displacing a maximum when it is itself nearly zero, or more accurately when its differential coefficient relatively to $\omega$ is at a maximum. As $\omega$ increases by $2\pi$ so as to pass from one principal maximum of oscillation to another, $\delta \omega$ passes $R$ times through $2\pi$, so that the term passes through as many maxima and minima. Then the differential coefficient relative to $\omega$ of the sum of all the terms will be the greatest for a value of $\omega$ such that
\[
\omega + m\theta = 2N\pi,
\]
[where $m$ being a given value of $m$], when, in addition to the above equation, we have
\[
R\theta = 4N\pi.
\]

In this case, the differential coefficient of the $n$th term of the expression for the oscillation will be
\[
\frac{R}{\omega} \left(\frac{\sin^2 n\theta}{2\pi} \frac{1}{\omega + m\theta}\right),
\]

It will be sufficiently accurate to put
\[
\sin \frac{1}{2} (\omega + m\theta) \approx \frac{1}{2} (m - m_0) \theta.
\]

Then it is plain that, were the term for $m = 0$ of the same value as the others, the total differential coefficient would be
\[
\frac{R}{\omega} m_0 e^{\gamma},
\]

Owing, however, to the term for $m = 0$ having only half the value given by the formula, the value is
\[
\frac{R}{\omega} m_0 e^{\gamma} - \frac{1}{2}.
\]

In consequence of the differential coefficient having this value, the maximum will not occur exactly at the value of $\omega$ for which
\[
\omega + m\theta = 2N\pi,
\]

but will be shifted along to the point where the differential coefficient of the $n$th term is equal to the negative of the differential coefficient just found. If $\delta \omega$ is the amount of the shifting, the $n$th term of the oscillation ($R$ being very large) is
\[
\sin \frac{R}{\omega} \delta \omega.
\]

The differential coefficient of this is
\[
\frac{1}{4} \sin R \omega - R \omega
\]
and the equation to determine $\delta \omega$ is
\[
\frac{1}{4} \frac{\sin R \omega - R \omega}{(\omega)^2} = \frac{R}{\omega} m_0 (e^{\gamma} - 1).
\]

In the worst case, this becomes
\[
\delta \omega = 24 \frac{m_0}{R} (\gamma^2 - 1).
\]

It thus appears that the position of the principal spectrum will not be disturbed by the circumstance here considered, and that the distance between the successive ghosts will be very slightly altered.

It is to be remarked that, when two spectral lines fall very near together, they will be attracted to one another in consequence of the mixture of light
by a sensible amount. This will especially affect the position of a faint line near a very intense one.

**The Phenomena.**

Mr. Rutherford's diffraction-plates are ruled with a machine which is described by Professor A. M. Mayer in the article "Spectrum," in the second edition of *Appleton's Cyclopedia*. In consequence of the periodic error of the screw, a periodic inequality is produced in the ruling. This is shown by putting a gitter into the spectrometer, illuminating it with homogenous light, and observing it without the eye-piece, when it appears striped. If the eye-piece is replaced and a real solar spectrum is thrown on the slit-plate, of such purity that the light admitted into the slit varies only by a few ten-thousandths of a micron in wave-length, the maxima of light which have been investigated above appear as repetitions of the principal spectrum, in which even the fine lines due to the solar atmosphere are distinctly visible.

The positions of some of these "ghosts," or repetitions of the principal spectrum, have been carefully measured in order to test the theory.

**Measure of the Positions of the Ghosts.**

To determine whether the screw of the filar micrometer had the same pitch throughout its length, the distance between D1 and D2 was measured on different places on the screw. Gitter: speculum metal 681 lines to the millimeter. Second order, principal spectrum. Readings given are mean of five readings each. Date: 1870, July 3.

| Line of Spectrum | First End. | Second End. | Second End. | First End. | Distance of Lines
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<td>D3</td>
<td>D4</td>
<td>D5</td>
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<tr>
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<td>D5</td>
<td>0.533</td>
</tr>
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</table>

The following were made with a speculum-metal gitter of 3490 teeth to the millimeter. Each reading given is the mean of five readings. Date: 1870, July 3. To pass from one spectrum to another the gitter alone was turned.

<table>
<thead>
<tr>
<th>Order of Spectrum</th>
<th>Number of Ghost</th>
<th>Ghost, — 1</th>
<th>Order IV</th>
<th>Order V</th>
<th>Ghost, 0</th>
<th>Ghost, + 1</th>
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<td>D4</td>
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<td>1.472</td>
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<td>1.471</td>
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<td>1.468</td>
<td>1.472</td>
<td>1.472</td>
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Mean...
The following measures were made with a metal gitter of 681 lines to the millimeter. Dates: 1879, June 20 and July 2.

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<th>Ghost, 5</th>
<th>Ghost, 6</th>
<th>Ghost, 7</th>
<th>Ghost, 8</th>
<th>Ghost, 9</th>
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<th>Ghost, 12</th>
<th>Ghost, 13</th>
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<th>Ghost, 15</th>
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<td>D₁, D₂, D₃, D₄</td>
<td>D₁, D₂, D₃, D₄</td>
<td>D₁, D₂, D₃, D₄</td>
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<td>D₁, D₂, D₃, D₄</td>
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<td>D₁, D₂, D₃, D₄</td>
<td>D₁, D₂, D₃, D₄</td>
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<td>D₁, D₂, D₃, D₄</td>
<td>D₁, D₂, D₃, D₄</td>
<td>D₁, D₂, D₃, D₄</td>
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<tr>
<td>D₀</td>
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</tr>
</tbody>
</table>

The following measures were made on spectra produced by a narrow sliver of light plate of 681 lines to the millimeter. This grating was selected as making unusually bright ghosts. The diffraction by the grating must specially displace the ghosts. The two horizontal lines and the inclined line between them, were observed. Date: 1879, June 20.
The following measures were made upon C, with the metal gitter of 681 lines per mm. The distance of the fine line 3 = 6567.91 (Å) from C was measured in the principal spectrum to determine the dispersion. Date: 1870, July 1.

<table>
<thead>
<tr>
<th>Order I</th>
<th>Ghost, — 1</th>
<th>Ghost, 0</th>
<th>Ghost, + 1</th>
<th>Fine line</th>
<th>C.</th>
<th>C.</th>
<th>C.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.551</td>
<td>1.407</td>
<td></td>
<td>0.546</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Order II.**

<table>
<thead>
<tr>
<th>Ghost, — 1</th>
<th>Ghost, 0</th>
<th>Ghost, + 1</th>
<th>Fine line</th>
<th>C.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.887</td>
<td>1.833</td>
<td></td>
<td>1.331</td>
</tr>
</tbody>
</table>

**Order III.**

<table>
<thead>
<tr>
<th>Ghost, — 1</th>
<th>Ghost, 0</th>
<th>Ghost, + 1</th>
<th>Fine line</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.718</td>
<td>10.210</td>
<td>12.784</td>
<td>7.004</td>
</tr>
<tr>
<td></td>
<td>2.995</td>
<td>2.724</td>
<td></td>
</tr>
</tbody>
</table>

The following measure was made upon F, with the same gitter. The mean of lines 4870.47 and 4871.20 was pointed on to determine the dispersion. Date: 1870, July 1.

**Order II.**

<table>
<thead>
<tr>
<th>Double</th>
<th>Ghost, 0</th>
<th>Ghost, 0</th>
<th>Ghost, + 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.617</td>
<td>10.740</td>
<td>11.683</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.867</td>
<td>1.190</td>
<td></td>
</tr>
</tbody>
</table>

The above measures satisfy the theory moderately well. Thus, according to theory, the product of the ratio of the distance of successive ghosts to the distance between the D line by the order of the spectrum should be constant, and should be twice as great for the gitter of 3401 lines to the millimeter as for that of 681 lines to the millimeter. Now this product is as follows:

**Metal Gitter of 3401 lines to the mm.**

<table>
<thead>
<tr>
<th>Order</th>
<th>5.43 = 2 x 2.73</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>5.44 = 2 x 2.73</td>
</tr>
<tr>
<td>VI</td>
<td>5.55 = 2 x 2.73</td>
</tr>
<tr>
<td>VII</td>
<td>5.59 = 2 x 2.73</td>
</tr>
<tr>
<td>VIII</td>
<td>5.65 = 2 x 2.73</td>
</tr>
<tr>
<td>IX</td>
<td>5.66 = 2 x 2.73</td>
</tr>
</tbody>
</table>
Metal Gitter of 681 lines to the mm.

<table>
<thead>
<tr>
<th>Order</th>
<th>Obs.</th>
<th>Calc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>2.75</td>
<td>2.75</td>
</tr>
<tr>
<td>II.</td>
<td>2.74</td>
<td>2.74</td>
</tr>
<tr>
<td>III.</td>
<td>2.74</td>
<td>2.74</td>
</tr>
<tr>
<td>IV.</td>
<td>2.74</td>
<td>2.74</td>
</tr>
</tbody>
</table>

Silvered-glass Gitter of 681 lines to the mm.

<table>
<thead>
<tr>
<th>Order</th>
<th>Obs.</th>
<th>Calc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>2.68</td>
<td>2.68</td>
</tr>
<tr>
<td>II.</td>
<td>2.74</td>
<td>2.74</td>
</tr>
<tr>
<td>III.</td>
<td>2.74</td>
<td>2.74</td>
</tr>
<tr>
<td>IV.</td>
<td>2.74</td>
<td>2.74</td>
</tr>
</tbody>
</table>

It is evident that the value which best satisfies the observations lies between 2.74 and 2.75. This ratio multiplied by the ratio of the difference of the D lines to their mean wave-length, should give the number of lines of the finer gitter to a period of the inequality. This, from the construction of the ruling-machine, is known to be nearly, but not exactly, 300. Mr. Chapman, who works with the machine, has made certain observations from which it would appear that the period differs about 1 per cent. from 300. The product of the ratios just mentioned (taking 2.746 for the first) is 367. This is therefore a happy confirmation of the theory.

Next, using the value 2.746, I calculate by least squares the best values of the distance of the D lines and the distance of consecutive ghosts in each order. In this way, we shall be able to judge whether the discrepancies of the observations from theory are, or are not, greater than their probable errors. The results are as follows:

<table>
<thead>
<tr>
<th>Distance $D_4 - D_1$</th>
<th>Distance of successive Ghosts</th>
</tr>
</thead>
<tbody>
<tr>
<td>IV.</td>
<td>1.084</td>
</tr>
<tr>
<td>V.</td>
<td>1.480</td>
</tr>
<tr>
<td>VI.</td>
<td>2.036</td>
</tr>
<tr>
<td>VII.</td>
<td>2.936</td>
</tr>
<tr>
<td>VIII</td>
<td>4.679</td>
</tr>
<tr>
<td>IX.</td>
<td>12.532</td>
</tr>
</tbody>
</table>

Detailed Comparison of Calculation and Observation.

<table>
<thead>
<tr>
<th>Metal Gitter of 3404 lines per mm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order IV.</td>
</tr>
<tr>
<td>Obs.</td>
</tr>
<tr>
<td>G - 1, D_1</td>
</tr>
<tr>
<td>G - 1, D_2</td>
</tr>
<tr>
<td>G - 1, D_3</td>
</tr>
<tr>
<td>G - 1, D_4</td>
</tr>
<tr>
<td>G + 1, D_1</td>
</tr>
<tr>
<td>G + 1, D_2</td>
</tr>
</tbody>
</table>
Order V.

$G = 1, \theta_1 = 3.847$  \hspace{0.5cm} 3.846  \hspace{0.5cm} +0.001 \hspace{0.5cm} $Carried toward G0, D_3.$

$G = 1, \theta_1 = 3.337$  \hspace{0.5cm} 3.332  \hspace{0.5cm} +0.000 \hspace{0.5cm} $Carried toward G1, D_3.$

$G = 1, \theta_1 = 9.466$  \hspace{0.5cm} 9.472  \hspace{0.5cm} +0.006 \hspace{0.5cm} $Carried toward G(1 - \sqrt{3})D_3.$

$G = 1, \theta_1 = 10.262$  \hspace{0.5cm} 10.363  \hspace{0.5cm} +0.009 \hspace{0.5cm} $Carried toward G + 1, D_3.$

$G + 1, \theta_1 = 11.099$  \hspace{0.5cm} 11.088  \hspace{0.5cm} -0.008 \hspace{0.5cm} $Carried toward G0, D_3.$

$G + 1, \theta_1 = 12.570$  \hspace{0.5cm} 12.379  \hspace{0.5cm} +0.004 \hspace{0.5cm} $Carried toward G1, D_3.$

Order VI.

$G = 1, \theta_1 = 7.388$  \hspace{0.5cm} 7.388  \hspace{0.5cm} -0.001 \hspace{0.5cm}$Single pointings discordant. Repeating work.

$G = 1, \theta_1 = 9.293$  \hspace{0.5cm} 9.390  \hspace{0.5cm} +0.005 \hspace{0.5cm} $Carried toward G - 1, D_3.$

$G = 1, \theta_1 = 9.421$  \hspace{0.5cm} 9.433  \hspace{0.5cm} +0.012 \hspace{0.5cm} $Carried toward G0, D_3.$

$G + 1, \theta_1 = 11.152$  \hspace{0.5cm} 11.132  \hspace{0.5cm} +0.020 \hspace{0.5cm} $Carried toward G0, D_3.$

$G = 1, \theta_1 = 12.194$  \hspace{0.5cm} 12.093  \hspace{0.5cm} +0.009 \hspace{0.5cm} $Carried toward G + 1, D_3.$

$G + 1, \theta_1 = 13.172$  \hspace{0.5cm} 12.377  \hspace{0.5cm} +0.004 \hspace{0.5cm} $Carried toward G - 1, D_3.$

Order VII.

$G = 1, \theta_1 = 6.646$  \hspace{0.5cm} 6.646  \hspace{0.5cm} -0.009 \hspace{0.5cm} $Single pointings discordant. Repeating work.

$G = 1, \theta_1 = 8.364$  \hspace{0.5cm} 8.361  \hspace{0.5cm} +0.005 \hspace{0.5cm} $Carried toward G - 1, D_3.$

$G = 1, \theta_1 = 9.526$  \hspace{0.5cm} 9.582  \hspace{0.5cm} +0.013 \hspace{0.5cm} $Carried toward G0, D_3.$

$G + 1, \theta_1 = 11.362$  \hspace{0.5cm} 11.256  \hspace{0.5cm} +0.006 \hspace{0.5cm} $Carried toward G0, D_3.$

$G + 1, \theta_1 = 11.878$  \hspace{0.5cm} 11.867  \hspace{0.5cm} +0.011 \hspace{0.5cm} $Carried toward G + 1, D_3.$

$G + 1, \theta_1 = 14.391$  \hspace{0.5cm} 14.132  \hspace{0.5cm} +0.004 \hspace{0.5cm} $Carried toward G - 1, D_3.$

Order VIII.

$G = 1, \theta_1 = 4.737$  \hspace{0.5cm} 4.771  \hspace{0.5cm} +0.064 \hspace{0.5cm} $No distinct attractions.

$G = 1, \theta_1 = 8.002$  \hspace{0.5cm} 7.992  \hspace{0.5cm} +0.010 \hspace{0.5cm} $No distinct attractions.

$G = 1, \theta_1 = 9.467$  \hspace{0.5cm} 9.462  \hspace{0.5cm} +0.005 \hspace{0.5cm} $No distinct attractions.

$G = 1, \theta_1 = 11.256$  \hspace{0.5cm} 11.213  \hspace{0.5cm} +0.035 \hspace{0.5cm} $No distinct attractions.

$G = 1, \theta_1 = 15.688$  \hspace{0.5cm} 15.688  \hspace{0.5cm} +0.000 \hspace{0.5cm} $No distinct attractions.

$G = 1, \theta_1 = 15.685$  \hspace{0.5cm} 15.201  \hspace{0.5cm} +0.036 \hspace{0.5cm} $No distinct attractions.

Order IX.

$G = 1, \theta_1 = 5.883$  \hspace{0.5cm} 5.882  \hspace{0.5cm} +0.005 \hspace{0.5cm} $No distinct attractions.

$G = 1, \theta_1 = 4.261$  \hspace{0.5cm} 4.259  \hspace{0.5cm} +0.002 \hspace{0.5cm} $No distinct attractions.

$G = 1, \theta_1 = 9.406$  \hspace{0.5cm} 9.397  \hspace{0.5cm} +0.009 \hspace{0.5cm} $No distinct attractions.

$G + 1, \theta_1 = 12.048$  \hspace{0.5cm} 12.048  \hspace{0.5cm} +0.002 \hspace{0.5cm} $No distinct attractions.

$G + 1, \theta_1 = 16.937$  \hspace{0.5cm} 16.945  \hspace{0.5cm} +0.002 \hspace{0.5cm} $No distinct attractions.

$G + 1, \theta_1 = 24.435$  \hspace{0.5cm} 24.053  \hspace{0.5cm} +0.088 \hspace{0.5cm} $No distinct attractions.