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10. Limb wavy; star faint; first distinctly seen 1:5 after recorded time. Power 141.
20. Stars about 1' apart; northern and fainter first recorded.
36. Perhaps 9' or 9'6' early. Star faint, limb blinding.
46. Well observed; 9'6' too late; this correction has been applied.
56. Well observed; 9'6' too late; this correction has been applied.
66. 2 signals, 19' apart, reappearance estimated to precede last signal as much as 25 follows it.
76. First seen 19'6' after recorded time; distance from limb doubled 3', and tripled 19'4 after first appearance.
86. First contact of ball. Contacts of ring too near those of ball to observe.
116. Disappearance of ring B, inner edge. At least 9' late.
126. Disappearance of ring B, outer edge. Much too early, perhaps 10'.
136. Last contact of ring B, inner edge. Much too early, perhaps 10'.
146. Last contact of ring A, outer edge. Precisely well observed.
156. First contact of ring A. Close to terminator; seeing bad.
166. Moon low; planet faint.
Brief Description of the Algebra of Relatives.

By C. S. Peirce.

Let \( A, B, C, \text{etc.} \), denote objects of any kind. These letters may be conceived to be finite in number or innumerable. The sum of them, each affected by a numerical coefficient (which may equal 0), is called an absolute term. Let \( z \) be such a term; then we write

\[
z = (x)_n A + (x)_n B + (x)_n C + \text{etc.} = \Sigma (x)_n I.
\]

Here \((x), \text{etc.}\), are numbers, which may be permitted to be imaginary or restricted to being real or positive, or to being roots of any given equation, algebraic or transcendental.¹ By \(\phi z\), any mathematical function of the absolute term \(z\), we mean such an absolute term that

\[
(\phi z) = \phi (x).
\]

That is, each numerical coefficient of \(\phi z\) is the function, \(\phi\), of the corresponding coefficient of \(z\). In particular,

\[
(x + y)_n = (x)_n + (y)_n,
\]

\[
(x \times y)_n = (x)_n \times (y)_n.
\]

Otherwise written,

\[
z + y = \{(x)_n, (y)_n\} A + \{(x)_n, (y)_n\} B + \text{etc.}
\]

\[
z \times y = \{(x)_n, \times (y)_n\} A + \{(x)_n, \times (y)_n\} B + \text{etc.}
\]

Two peculiar absolute terms are suggested by the logic of the subject. I call them terms of second intention. The first is zero, 0, and is defined by the equation

\[
(0)_n = 0
\]

or

\[
0 = 0A + 0B + 0C + \text{etc.}
\]

¹ I have usually restricted the coefficients to one or other of two values; but the more general view was distinctly recognized in my paper of 1879.
The other is cxx (or non-relative unity), 0, and is defined by the equation

\[(0)_y = 1,\]

or

\[0 = A + B + C + \text{etc.}\]

The symbol \((A : B)\) is called an individual dual relation. It signifies simply a pair of individual objects, \((A : B)\) and \((B : A)\) being different. An aggregate of such symbols, each affected by a numerical coefficient, is called a general dual relation. The totality of pairs of letters arrange themselves with obvious naturalness in the block,

\[
\begin{align*}
A : A & \quad A : B & \quad A : C & \quad \text{etc.} \\
B : A & \quad B : B & \quad B : C & \quad \text{etc.} \\
C : A & \quad C : B & \quad C : C & \quad \text{etc.} \\
\end{align*}
\]

If \(I\) denotes any general dual relative, then the coefficient of the pair \(I : J\) in \(I\) is written \((I)_I\). These coefficients are thus each referred to a place in the above block, and may themselves be arranged in the block:

\[
\begin{align*}
(I)_A & = (I)_B & = (I)_C & = \text{etc.} \\
(I)_A & = (I)_B & = (I)_C & = \text{etc.} \\
(I)_A & = (I)_B & = (I)_C & = \text{etc.} \\
\end{align*}
\]

Every relative term, \(z\), is separable into a part called 'self'; \(S\), such that

\[S = S_z = (x)_z (I : J)\]

and the remaining part, called 'allo'; \(V_x\), comprising all the terms in \(x\) not in the principal diagonal of the block, so that we write

\[z = S_z + V_z\]

Each absolute term is considered to be equivalent to a certain relative term; namely,

\[A = (A : A) + (A : B) + (A : C) + \text{etc.}\]

or, if \(z\) be an absolute term,

\[(x)_z = (x)_y = 0.
\]

The self-part of the relative equivalent to an absolute term is denoted by writing a comma after the term. Accordingly,

\[(x)_y = (x)_z = 0.
\]

Besides 0 and 0, two other dual relative terms have been called terms of second intention. These are simply \(\bar{0}\) and \(\bar{y}\). The relative \(\bar{0}\) or \((\bar{0})_1\) is also written \(1\), and is called unity, or 'identical with.' It is defined by the equations

\[
\begin{align*}
(1)_L & = 1, & (1)_L & = 0. \\
1 & = (A : A) + (B : B) + (C : C) + \text{etc.}
\end{align*}
\]

The relative \(\bar{y}\) is written \(\bar{y}\) or \(\bar{y}\), and is called 'not,' or 'the negative of.' It is defined by the equations

\[
\begin{align*}
(\bar{y})_y & = 0, & (\bar{y})_y & = 1.
\end{align*}
\]

By an absolute function of a relative term is meant that function taken according to the rules for taking the function of an absolute term. That is,

\[
(\phi(x))_y = \phi(x)_y.
\]

In particular,

\[
\begin{align*}
(x + y)_y & = (x)_y + (y)_y \\
(x \times y)_y & = (x)_y \times (y)_y
\end{align*}
\]

Of the various external or relative combinations that have been employed the following may be particularly specified. (1), External multiplication, defined by the equation

\[(x \times y)_y = \Sigma (x)_z (y)_z.
\]

(2), External progressive involution, defined by the equation

\[(x^2)_y = \Pi (x)^2_y.
\]

(3), External regressive involution, defined by the equation

\[(x^2)_y = \Pi (y)^2_y.
\]

In general, using Miss Ladd's notation * for the different orders of multiplication,

\[(x \times y)_y = \Pi (x)_z (y)_z.
\]

Other modes of external combination have been used, but they are believed to have only a special utility. Division does not generally yield an unambiguous quotient. Indeed, I have shown that it does so only in the cases of ordinary real algebras, of imaginary algebras, and of real quadratures.

Besides the mathematical functions of relatives, there are various modes in

Algebra of Relatives.

which one relative may logically depend upon another. Thus, $Sx$ and $Fr$ may be said to be logical functions of $x$. The most important of such operations is that of taking the converse of a relative. The converse of $x$, written $\tilde{x}$ or $xx$, is defined by the equation

$$ (\tilde{x})y = (x)y. $$

The algebraical laws of all these combinations are obtained with great facility by a method of which the following are examples:

**Example 1.**

$$ [(xy)z]y = \Sigma_{y}(xy)z \Sigma_{z}(z)y = \Sigma_{y}(xy) \Sigma_{z}(z) $$

$$ = \Sigma_{y}(xy)z \Sigma_{z}(z)y. $$

$$ (xy)z = x(y)z. $$

**Example 2.**

$$ [x+y]z = \Sigma_{y}(x+y)z \Sigma_{z}(z) = \Sigma_{y}(x)z + \Sigma_{y}(y)z \Sigma_{z}(z) $$

$$ = \Sigma_{y}(x)z + \Sigma_{y}(y)z \Sigma_{z}(z) = (x)z + (y)z \Sigma_{z}(z). $$

$$ (x+y)z = xz + yz. $$

The following are some of the elementary formulas so obtained. Non-relative multiplication is indicated by a comma, relative multiplication by writing the factors one after the other, without the intervention of any sign.

$$ (x+y)z = xz + yz, $$

$$ (x,y,z) = x(y,z), $$

$$ x,y,z = (x,y,z), $$

$$ (xy)z = x(yz). $$

The multiplication of triple relatives is not perfectly associative and the multiplication of two triple relatives yields a quadruple relative.

The modes of combination of a triple relative followed by two double relatives are the same as the modes of combination of three double relatives. This ceases to be true for quadruple and higher relatives.

Corresponding to the operation of taking the converse of a relative, there are five operations upon triple relatives. They are defined as follows:

$$ (k_{1}k_{2}k_{3}) = (k_{1}k_{2})k_{3}, $$

$$ (k_{1}k_{2}) = (k_{1})(k_{2}), $$

$$ k_{1}k_{2}k_{3} = k_{3}k_{2}k_{1}. $$

The multiplication of quadruple or higher relative may be conceived as a product of triple relatives.

Thus, the essential characteristics of this algebra are (1) that it is a multiple algebra depending upon the addition of square blocks or cubes of numbers, (2) that in the external multiplication the rows of the block of the first factor are respectively multiplied by the columns of the block of the second factor, and (3) that the multiplication so resulting is for the two-dimensional form of the algebra, always associative. I have proved in a paper presented to the American Academy of Arts and Sciences, May 11, 1875, that this algebra necessarily embraces every associative algebra.

I have here described the algebra apart from the logical interpretation with which it has been clothed. In this interpretation a letter is regarded as a name applicable to one or more objects. By a name is usually meant something representative of an object to a mind. But I generalize this conception and regard
Algebra of Relatives.

a name as merely something in a conjoint relation to a second and a third, that is as a triple relative. A sum of different individual names is a name for each of the things named severally by the aggregate letters. A name multiplied by a positive integral coefficient is the aggregate of so many different senses in which that name may be taken. The individual relative $A : B$ is the name of $A$ considered as the first member of the pair $A : B$. The signification of the external multiplication is then determined by its algebraic definition.

Professor Sylvester, in his "New Universal Multiple Algebra," appears to have come, by a line of approach totally different from mine, upon a system which coincides, in some of its main features, with the Algebra of Relatives, as described in my four papers upon the subject,* and in my lectures on logic. I am unable to judge, from my unprofessional acquaintance with pure mathematics, how much of novelty there may be in my conceptions; but as the researches of the illustrious geometer who has now taken up the subject must draw increased attention to this kind of algebra, I take occasion to redescribe the outlines of my own system, and at the same time to declare my modest conviction that the logical interpretation of it, far from being in any degree special, will be found a powerful instrument for the discovery and demonstration of new algebraical theorems.

BOSTON, Jan. 7, 1862.

Postscript.—I have this day had the delight of reading for the first time Professor Cayley's Memoir on Matrices, in the Philosophical Transactions for 1858. The algebra he there describes seems to me substantially identical with my own subsequent algebra for dual relatives. Many of his results are limited to the very exceptional cases in which division is a determinative process.

My own studies in the subject have been logical not mathematical, being directed toward the essential elements of the algebra, not toward the solution of problems.

JANUARY 14, 1862.