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$(x_3, y_3, z_3)$ of the third point of intersection may be expressed indifferently in the two forms

$$x_3 : y_3 : z_3 = P : Q : R \text{ and } x_3 : y_3 : z_3 = A : B, C.$$

But these considered irrespectively of the equations $U_1 = 0$, $U_2 = 0$, are distinct formulae, each of them separately establishing a correspondence between the three points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$ or if we regard one of these points as a fixed point, then a correspondence between the remaining two points, or if we consider these as belonging each to its own plane, then a correspondence between two planes. Writing for convenience $(a, b, c)$ for the coordinates of the fixed point, and $(x_1, y_1, z_1)$ $(x_2, y_2, z_2)$ for those of the other two points, the formulae with $A, B, C$ give thus the correspondence

$$x_2 : y_2 : z_2 = bcx_1^2 - a^2y_1z_1 : cay_1^2 - b^2z_1x_1 : abz_1^2 - c^2x_1y_1,$$

which is the first of the two cases in question. These equations give reciprocally

$$x_1 : y_1 : z_1 = bcx_2^2 - a^2y_2z_2 : cay_2^2 - b^2z_2x_2 : abz_2^2 - c^2x_2y_2,$$

or the correspondence is a (1, 1) quadric correspondence.

The formulae with $P, Q, R$ give in like manner

$$x_2 : y_2 : z_2 = a(ax_1^2 + by_1^2 + cz_1^2) - x_1(a^2x_1 + b^2y_1 + c^2z_1), \&c.,$$

or if for shortness

$$\Omega_1 = ax_1^2 + by_1^2 + cz_1^2, \quad \Theta_1 = a^2x_1 + b^2y_1 + c^2z_1,$$

then

$$x_2 : y_2 : z_2 = a\Omega_1 - x_1\Theta_1 : b\Omega_1 - y_1\Theta_1 : c\Omega_1 - z_1\Theta_1,$$

which is the second of the two cases. We have reciprocally

$$x_1 : y_1 : z_1 = a\Omega_2 - x_2\Theta_2 : b\Omega_2 - y_2\Theta_2 : c\Omega_2 - z_2\Theta_2,$$

where $\Omega_2 = ax_2^2 + by_2^2 + cz_2^2$, $\Theta_2 = a^2x_2 + b^2y_2 + c^2z_2$, and the correspondence is thus in this case also a (1, 1) quadric correspondence.

## On a Question in Partitions, by J. J. SYLVESTER.

Closely connected with the theory of the contacts or special intersections of quadric figures in space of any number of dimensions, and also with the more general but allied theory of the different genera and species of the roots of unitary matrices, is the question of the number of series that can be formed commencing with zero and ending with a given number $i$ subject to the condition that each intermediate term of any such series shall be not greater than the mean between its antecedent and consequent. By arranging each of the indefinite Partitions of $i$ according to an ascending order of magnitude, it was shown that there was a one to one correspondence between each such arrangement and each such series, and, consequently, that the number of the series is equal to the number of indefinite partitions of the given final term $i$.

## On the Relative Forms of Quaternions, by C. S. PEIRCE.

If $X, Y, Z$ denote the three rectangular components of a vector, and $W$ denote numerical unity (or a fourth rectangular component, involving space of four dimensions), and $(Y : Z)$ denote the operation of converting the $Y$ component of a vector into its $Z$ component, then

$$\begin{aligned} 1 &= (W : W) + (X : X) + (Y : Y) + (Z : Z) \\ i &= (X : W) - (W : X) - (Y : Z) + (Z : Y) \\ j &= (Y : W) - (W : Y) - (Z : X) + (X : Z) \\ k &= (Z : W) - (W : Z) - (X : Y) + (Y : X). \end{aligned}$$

In the language of logic $(Y : Z)$ is a relative term whose relate is a $Y$ component, and whose correlate is a $Z$ component. The law of multiplication is plainly $(Y : Z)(Z : X) = (Y : X)$, $(Y : Z)(X : W) = 0$, and the application of these rules to the above values of $1, i, j, k$ gives the quaternion relations

$$i^2 = j^2 = k^2 = -1, \; ijk = -1, \&c.$$

The symbol $a(Y : Z)$ denotes the changing of $Y$ to $Z$ and the multiplication of the result by $a$. If the relatives be arranged in the block

$$\begin{matrix} W:W & W:X & W:Y & W:Z \\ X:W & X:X & X:Y & X:Z \\ Y:W & Y:X & Y:Y & Y:Z \\ Z:W & Z:X & Z:Y & Z:Z, \end{matrix}$$

then the quaternion $w + xi + yj + zk$ is represented by the matrix of numbers

$$\begin{matrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w. \end{matrix}$$

The multiplication of such matrices follows the same laws as the multiplication of quaternions. The determinant of the matrix = the fourth power of the tensor of the quaternion.

The imaginary $x + y\sqrt{-1}$ may likewise be represented by the matrix

$$\begin{matrix} x & y \\ -y & x, \end{matrix}$$

and the determinant of the matrix = the square of the modulus.

## On a Geometrical Proof of a Theorem in Numbers, by J. J. SYLVESTER.

The theorem in question is the well-known one that if $a$, $b$ are incom- mensurable and $x, y$ integers $ax + by + c$ may be made positively an[d] negatively indefinitely small. This is tantamount to showing that on th[e] plane of a reticulation,* nodes may be found indefinitely near to and o[n] each side of an irrational straight line, $i. e.$ a line not parallel to any lin[e] of nodes. The proof is based on the Lemma that no infinite parallelogram each side of which is an irrational line containing a node, can be vacuou[s] of nodes in its interior. If this were not true a succession of shifts of th[e] figures in the direction of the line forming the two nodes would lead to th[e] absurd conclusion that the whole reticulation consists of a single line of node[s].

1°. Suppose the irrational line $L$ contains a node and that there is n[o] other node at less than a finite distance from it on one side of it, say t[o] the right. Let it be moved to the right parallel to itself until it pass[es] through another node $N'$, then there will be a vacuous parallelogram o[f] the kind declared impossible by the Lemma. [To this it may be objecte[d] that when $L$ has moved from the left to $M$ through a distance $\delta$, $M$ migh[t] be supposed to be an asymptote to an infinite series of nodes to its righ[t]. But if this were the case a node $P$ might be found at a less distance tha[n] $\delta$ from $M$, and a node, $Q$, nearer to $M$ than $P$ is; if this line of node[s] $PQ$ be followed up until we reach the first node $T$ on the other side of $M$, th[e] most elementary geometry seems to show that $T$ in *any* case is nearer t[o] $M$ than $P$ is and consequently there would be a node between $L$ and [M] contrary to hypothesis.] Hence there must be a node indefinitely nea[r] to $L$ on each side of it.

2°. Suppose the irrational line $L$ not to contain a node. If the theore[m] to be proved is not true, $L$ may as before be moved parallel to itsel[f] (through a finite distance) until it pass through a single node and ther[e] would be a vacuous parallelogram of which one side contains nodes, whic[h] has already been shown to be impossible.

Dr. Story and Dr. Franklin took part in the discussion and the valuabl[e] critical observations of the latter, led to the consideration of the objectio[n] stated and disposed of in the passage within brackets above. Professor Cay- ley made a remark to the effect that the diamond point in a graver's too[l] however fine, drawn in a straight direction across the face of a double gra[t]- ing must either pass through none of the intersections of the two systems o[f] parallel lines or through an infinite number of them. The principle estab- lished in the bracketed passage admits of being stated in the followin[g] terms: "It is impossible for a straight line in the plane of a reticulatio[n] to be asymptotic in regard to nodes on one side of it and not so in regard t[o] the nodes on the other side;" this proposition and the Lemma being con- ceded, the existence of any indefinite *vacuous* strip bounded by irration[al] parallel lines is disproved by imagining it distended on both sides, sti[ll] retaining its form (in case neither bounding line contains a node), or in th[e] contrary case on one side only ($i. e.$ in the direction away from the nod[al] line) until the distended figure passes through two nodes. The asymp- totic rule shows that this construction would be possible—the Lemma th[at] it leads to an impossible result. From this it follows that every irration[al] line is asymptotic in respect to the nodes lying on *each* side of it whic[h] is the thing to be proved.

Let a line be termed mono-asymptotic when it is asymptotic in regar[d] to any scheme of points lying on one side of it,—amphi-asymptotic whe[n] it is so for schemes of points lying on each side of it. The foregoing arg[u]- ment may then be summed up as follows. Any irrational right line i[n] the plane of a reticulation, must be amphi-asymptotic as regards th[e] nodes. For if not, a line parallel to it must (under pain of contradictin[g] the Lemma) be conceded to exist, which shall be mono-asymptotic i[n] respect to them, but the existence of such a line has been proved to b[e] impossible.† Similarly, it may be shown for a solid network, that no indefi- nite open prism whose parallel edges are doubly irrational ($i. e.$ neith[er] parallel to a nodal line nor to a plane of nodes) can be vacuous of node[s] and also that no plane can be mono-asymptotic—from which, by ver[y] similar reasoning to that previously used, may be deduced the law, th[at] no prism of finite dimensions, *vacuous* of nodes, can be constructed about a[n] irrational line as its axis and that consequently any such line may be regarde[d] as a sort of asymptotic axis to a helical spiral of nodes. Hence it follow[s] that if $a, b, c$ (taken two and two) are incommensurable with each othe[r] the quadratic function

$$(b(z-\gamma) - c(y-\beta))^2 + (c(x-a) - a(z-\gamma))^2 + (a(y-\beta) - b(a-a))^2$$

and as a particular case

$$(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2$$

may be made indefinitely small with integer values of $x, y, z$.

Nor is this all, for not only can a node be found indefinitely near to th[e] doubly irrational line $x : y : z :: a : b : c$, but such node may be successful[ly] sought for within any infinitesimal sector of space contained within tw[o] planes drawn through that line, or in other words a node can be foun[d]

* By a reticulation is to be understood a pair of systems of an infinite number of inde- nite equidistant parallel lines in a plane whose intersections form the nodes.

† The form of proof is a somewhat unusual combination of an *Ex-absurdo* with a *Dile- ma*. A *denial* of the amphi-asymptoticism of an irrational straight line either dash[es] itself against the impossibility of the existence of a vacuous parallelogram or against t[he] equal impossibility of the existence of a mono-asymptotic line.