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the Theory of Elliptic Functions.

If \( \phi u \) denotes any elliptic function of the 1st degree with the periods \( 2 \alpha, 2 \beta \), and if \( \phi u \) has the same pair of periods, then we can determine the 2\( r + 1 \) quantities

\[
\begin{align*}
&u_1, u_2, \ldots, u_r; \\
&v_1, v_2, \ldots, v_r; \\
&\phi(u) = C \phi(\alpha - u) \phi(\beta - u) \ldots \phi(u - \alpha - \beta); \\
&\phi(\alpha - u) \phi(\beta - u) \ldots \phi(u - \alpha - \beta),
\end{align*}
\]

which proposition is capable of inversion. An analogous theorem in regard to \( \psi u \) is, if

\[
\begin{align*}
&u_0, u_1, u_2, \ldots, u_n \\
&\psi(u_0, u_1, u_2, \ldots, u_n)
\end{align*}
\]

denote \( n + 1 \) independent variables, then the function

\[
\begin{align*}
&\psi(u_0, u_1, u_2, \ldots, u_n) = \\
&\begin{cases}
1 \mu_0 \nu_0^{2 \mu_0 - 2} \nu_0^{2 \nu_0 - 2} \\
1 \mu_0 \nu_0^{2 \mu_0 - 2} \nu_0^{2 \nu_0 - 2} \\
1 \mu_0 \nu_0^{2 \mu_0 - 2} \nu_0^{2 \nu_0 - 2} \\
\vdots \\
1 \mu_0 \nu_0^{2 \mu_0 - 2} \nu_0^{2 \nu_0 - 2}
\end{cases}
\end{align*}
\]

in an elliptic function of the degree \( n + 1 \) of any one of the arguments \( u_0, u_1, \ldots, u_n \). In general, every unique elliptic function \( \psi(\alpha) \) is expressible as a rational function of \( \phi u \) and the first derivative \( \phi u \) with the same pair of periods \( 2 \alpha, 2 \beta \) as \( \psi(u) \); and in like manner \( \phi u \) and \( \psi u \) are expressible as rational functions of \( \phi u \) and \( \psi u \).

With the function \( \phi u \) are closely connected the following

\[
\begin{align*}
&\phi u = \epsilon^{2 \alpha} \phi(\alpha + u) \\
&\phi u = \epsilon^{2 \alpha} \phi(\alpha + u) \\
&\phi u = \epsilon^{2 \alpha} \phi(\alpha + u) \\
&\phi u = \epsilon^{2 \alpha} \phi(\alpha + u)
\end{align*}
\]

where \( \alpha, \beta \) are the half-periods, and \( \alpha + \beta = \alpha \). By inserting in the "pocket edition" for the value of the \( \epsilon \) the values respectively \( \alpha, \beta, \alpha \), we have

\[
\begin{align*}
&\phi u = \epsilon^{2 \alpha} \phi(\alpha + u) \\
&\phi u = \epsilon^{2 \alpha} \phi(\alpha + u) \\
&\phi u = \epsilon^{2 \alpha} \phi(\alpha + u)
\end{align*}
\]

whereby the following relations are established for the differences of the roots. Remembering that \( \phi u = (\gamma, \beta, \alpha) \), \( \phi u = (\alpha, \beta, \gamma) \), \( \phi u = (\gamma, \alpha, \beta) \), we have

\[
\begin{align*}
&\sqrt{\beta - \alpha} = \frac{\gamma}{\alpha}, \\
&\sqrt{\gamma - \alpha} = \frac{\alpha}{\beta}, \\
&\sqrt{\alpha - \gamma} = \frac{\beta}{\gamma},
\end{align*}
\]

where we assume \( \alpha > \beta > \gamma \). If we now assume \( R \left( \frac{\alpha}{\beta} \right) > 0 \), that is, the real component of the complex \( \frac{\alpha}{\beta} > 0 \), that is, the geometrical representation of \( \frac{\alpha}{\beta} > 0 \), then

\[
\begin{align*}
&\phi u = (\gamma, \beta, \alpha) \\
&\phi u = (\alpha, \beta, \gamma) \\
&\phi u = (\gamma, \alpha, \beta)
\end{align*}
\]

satisfy the same differential equation

\[
\begin{align*}
&\left( \frac{d\phi u}{du} \right)^2 = (1 - (\alpha - \gamma) \epsilon)(1 - (\gamma - \alpha) \epsilon) \\
&\text{where} \\
&\left( \frac{d\phi u}{du} \right)^2 = (1 - (\alpha - \gamma) \epsilon)(1 - (\gamma - \alpha) \epsilon),
\end{align*}
\]

and the four functions

\[
\begin{align*}
&\phi u, \psi u, \psi u, \psi u, \\
&\phi u, \psi u, \psi u, \psi u, \\
&\phi u, \psi u, \psi u, \psi u
\end{align*}
\]

satisfy the relation

\[
\begin{align*}
&\left( \frac{d\phi u}{du} \right)^2 = (1 - (\alpha - \gamma) \epsilon)(1 - (\gamma - \alpha) \epsilon) \\
&\text{whereby the following relations are established for the differences of the roots.}
\end{align*}
\]

But the English reader will desire to know in what connection the system of Weierstrass stands to the more widely known systems of Jacobi and Legendre. If we define the \( k \) of Jacobi by the equation

\[
\frac{u}{u} = \frac{k}{1 - k^2}
\]

then the following relations are established between the sigma-quotients and Jacobi's functions. We give only three as specimens, replacing the \( \lambda, \mu, \nu \) by

\[
\begin{align*}
&\phi u, 1, \phi u, \\
&\phi u = \phi(\gamma, \alpha, \beta) \\
&\phi u = \phi(\beta, \gamma, \alpha) \\
&\phi u = \phi(\alpha, \beta, \gamma)
\end{align*}
\]

Not all the sigma-quotients are so nearly identical with Jacobi's functions, but in all cases the argument \( u \) appears multiplied with the same factor \( \sqrt{\beta - \alpha} \) which is the largest of the three root-differences.

In the defining equation \( P^2 = \frac{3}{4} - \frac{1}{2} \) and the corresponding one \( k^2 = \frac{3}{4} - \frac{1}{2} \), both of these quantities if real must be greater than zero and less than unity.
They will be real if the points in the plane representing $\omega_1$, $\omega_2$, $\omega_3$ lie in the same straight line, when mod. $\omega_1$ must be intermediate between mod. $\omega_2$ and mod. $\omega_3$ in magnitude. Then if we understand by $K$ and $K'$ the simplest values of $\gamma$ the integrals
\[
\int \frac{dx}{\sqrt{x - 1}}, \int \frac{dx}{\sqrt{x - 1}}
\]
respectively, taking those values of the radicals whose real components are positive, we shall have
\[
\omega_1 \sqrt{x - 1} = iK, \quad \omega_2 \sqrt{x - 1} = iK',
\]
and $2\omega_2, 2\omega_3$ are the primitive pair of periods for the before mentioned $\omega_1$, so that as above
\[
\omega_1 \omega_2 = \omega_2 \omega_3 = \omega_3 \omega_1 = \omega.
\]

It ought to be mentioned that $G_{\omega_1}, G_{\omega_2}, G_{\omega_3}$ can also be defined in the same simple manner as $G_{\omega}$ by means of infinite products. If we write
\[
\omega_1 = (2\omega + 1) \omega + 2\omega' \omega', \quad \omega_2 = (2\omega' + 1) \omega + (2\omega + 1) \omega', \quad \omega_3 = 2\omega + (2\omega' + 1) \omega',
\]
then in general, for $\lambda = 1, 2, 3,
\[
G_{\omega_\lambda} = e^{-\pi i \omega' \omega_{\lambda}} \left(1 - \frac{\omega_{\lambda}}{\omega_{\lambda - 1}}\right) e^{\frac{\pi i \omega_{\lambda - 1}}{\omega}}.
\]

Finally to show the relation in which the sigma functions stand to the $S$-functions of Jacobi, we find
\[
S_{\omega} = \frac{2\omega}{2\omega_1}, \quad S_{\omega_1} = \frac{2\omega_1}{2\omega_1}, \quad S_{\omega_2} = \frac{2\omega_2}{2\omega_2}, \quad S_{\omega_3} = \frac{2\omega_3}{2\omega_3},
\]
and
\[
G_{\omega} = e^{\pi i \omega' \omega} \left(1 - \frac{\omega}{\omega - 1}\right) e^{\frac{\pi i \omega}{\omega}}.
\]

The functions $S_{\omega}(\gamma), S_{\omega_1}(\gamma), S_{\omega_2}(\gamma), S_{\omega_3}(\gamma)$ as here employed coincide respectively with Jacobi's $S_{\omega}(\gamma), S_{\omega_1}(\gamma), S_{\omega_2}(\gamma), S_{\omega_3}(\gamma)$, if we write $\omega = \omega$ and $\omega = \omega$.

But anything more than a slight account of Weierstrass' system, showing in particular its main points of contact with Jacobi's, would be beyond the intention of this paper. It is to be hoped that Weierstrass' ideas in the function theory will soon find that widespread recognition which they undoubtedly merit. In a future paper I hope to exhibit the system in greater detail, in particular the formulæ of transformation, showing their analogies to the formulæ of Jacobi.
then, following Weierstrass and Schottky, and writing

\[ R(z) = (a_i - a_i)(a_i - a_i)(a_i - a_i)(a_i - a_i) \]

we have

1. \[ P_0 = 1 \]

2. \[ P_i = \sqrt{(a_i - a_i)(a_i - a_i)(a_i - a_i)(a_i - a_i)} \]

3. \[ P_n = P_0 P_1 \sum_{i=1}^{n} \sqrt[3]{R(z)} \]

4. \[ P_{m-n} = P_0 P_1 P_2 P_3 \sum_{i=1}^{n} \sqrt[3]{R(z)} \]

making in all 256 \( P \)-functions replacing the 256 \( \Theta \)-functions. In these equations the letters \( i, j, k, l \) and \( m, n, a \) are all different from each other. We have now to determine the linear relations existing between the squares of these \( P \)-functions and those existing between their products taken two and two. Write

\[ \Sigma a_i = a, \Sigma a_i a_j = \beta, \Sigma a_i a_j a_k = \gamma, a_i a_j a_k a_l = \delta. \]

The summations to be taken from 1 to 4 and \( i, j, k, l \) all having different values. Further write

\[ (a_i - a_i)(a_i - a_i)(a_i - a_i)(a_i - a_i) = -\theta, \]

that is

\[ \begin{array}{cccc}
| a_i | & a_i & a_i & a_i \\
\hline
| a_i | & 1 & 1 & 1 \\
\end{array} \]

and in general write

\[ \begin{array}{cccc}
| a_i | & a_i & a_i & a_i \\
\hline
| a_i | & 1 & 1 & 1 \\
\end{array} \]

that is

\[ | a_i a_j a_k a_l | = -\theta. \]

\[ \text{Annali di Matematica, Ser. II, Tomo 8.} \]

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\[ \text{Brioschi—Le relazioni di Gépel per funzioni iperellittiche d'ordini quindicimodo.} \]
The $P$-functions can now be written in the following manner,

$$P_1 = \sqrt{\sqrt{\frac{a_1 a_2}{a_3 a_4}}}$$

and

$$P_2 = \sqrt{\sqrt{\frac{a_5}{a_6 a_7}}}$$

The $P_{110}$ are given by

$$P_{110} = \frac{1}{|a_5 a_6 a_7|} \left| a_2 a_3 \right| \sqrt{B(\alpha)} \left| a_5 a_6 a_7 \right| \left| a_1 a_2 a_3 a_4 a_5 a_6 a_7 \right|$$

and

$$P_{110} = \frac{1}{|a_5 a_6 a_7|} \left| a_2 a_3 \right| \sqrt{B(\alpha)} \left| a_5 a_6 a_7 \right| \left| a_1 a_2 a_3 a_4 a_5 a_6 a_7 \right|$$

It is of course perfectly obvious how to fill up the empty radical signs. The above forms for the $P$-functions are retained for the same reason that Brunei gives in the case of the triple theta functions, that is, although the denominators under the radical signs are actually factors of $B(\alpha)$, $B(\beta)$, and $B(\gamma)$, it is more convenient in the following transformations to retain the fractional form.

**Linear Relations Between the Squares of the P-Functions.**

Take first the case of the functions with a single index, i.e., $P_1$, $P_2$, etc. We have

8. $P_2 = a_1 a_2 a_3 a_4 a_5 a_6 a_7$

    By expanding this and using the notation given above, we have

9. $P_1 = a_1 a_2 a_3 a_4 a_5 a_6 a_7$

As this is linear in the quantities $a$, $\beta$, $\gamma$, $\delta$, and as $k$ is any one of the primitive indices, we can, by assuming any five such relations, eliminate $a$, $\beta$, $\gamma$, and $\delta$; the result of the elimination is obviously

10. $P_1 = a_1 a_2 a_3 a_4 a_5 a_6 a_7$

    $a_1 a_2 a_3 a_4 a_5 a_6 a_7$

or expanding this we have

11. $P_1 = a_1 a_2 a_3 a_4 a_5 a_6 a_7 + P_2 a_1 a_2 a_3 a_4 a_5 a_6 a_7$

    $P_1 = a_1 a_2 a_3 a_4 a_5 a_6 a_7 + P_2 a_1 a_2 a_3 a_4 a_5 a_6 a_7$

Since $P_1 = 1$, this factor may be introduced solely for the sake of symmetry.

If instead of eliminating $a$, $\beta$, $\gamma$, and $\delta$ between five relations of the form (9), we eliminate $a$, $\beta$, $\gamma$, and $\delta$ between six such relations, we have obviously

$$P_1 = a_1 a_2 a_3 a_4 a_5 a_6 a_7 + P_2 a_1 a_2 a_3 a_4 a_5 a_6 a_7$$

and

$$P_1 = a_1 a_2 a_3 a_4 a_5 a_6 a_7 + P_2 a_1 a_2 a_3 a_4 a_5 a_6 a_7$$

or expanding

12. $P_1 = a_1 a_2 a_3 a_4 a_5 a_6 a_7 + P_2 a_1 a_2 a_3 a_4 a_5 a_6 a_7$

We have thus found the linear relations existing between the squares of the $P$-functions possessing a single suffix, or index, i.e., between the functions whose indices are $0$ $k$ $l$ $m$ $n$ $p$ $q$ $r$ $s$ $t$.

and it is seen that these functions form a group of ten such that any five being given the square of any one of the remaining five can be expressed as a linear function of the squares of the chosen five. Following Brunei I shall call this the group $0$.

Consider next the case of the $P$-functions with two suffixes: for the square of any one of them, say $P_{11}$, we have

13. $P_{11} = \frac{1}{|a_5 a_6 a_7|} \left| a_1 a_2 a_3 a_4 a_5 a_6 a_7 \right| R(\alpha) \left| a_1 a_2 a_3 a_4 a_5 a_6 a_7 \right|$

or expanding

14. $P_{11} = \frac{1}{|a_5 a_6 a_7|} \left| a_1 a_2 a_3 a_4 a_5 a_6 a_7 \right| R(\alpha) \left| a_1 a_2 a_3 a_4 a_5 a_6 a_7 \right|$

It is possible to find a linear relation between four of these $P$-functions with two suffixes which is entirely rational, that is, a relation which shall not contain any of the quantities $\sqrt{R(\alpha)} R(\alpha)$. Take four of the functions $P_{11}$ which have the first suffix $k$ in common, say $P_{11}$, $P_{11}$, $P_{11}$, $P_{11}$, then in order that the radicals $\sqrt{R(\alpha)} R(\alpha)$ may disappear we must find a series of multipliers $A$, $B$, $C$, $D$, satisfying the equation

15. $A |a_5 a_6 a_7| - B |a_5 a_6 a_7| + C |a_5 a_6 a_7| - D |a_5 a_6 a_7| = 0$.
Giving $A$, $B$, $C$ and $D$ the following values,

\[ A = [a_1, a_2, a_3], \quad B = [a_4, a_5, a_6] \]
\[ C = [a_7, a_8, a_9], \quad D = [a_10, a_11, a_12] \]

it is easy to see that equation 15 is satisfied. Assuming then four equations of the same form as 14, we have

\[
\frac{1}{[a_1, a_2, a_3]} \left[ \begin{array}{c} [a_4, a_5, a_6] \cdot [a_1, a_2, a_3] \\ [a_7, a_8, a_9] \cdot [a_4, a_5, a_6] \\ [a_0, a_1, a_2] \cdot [a_7, a_8, a_9] \\ \vdots \end{array} \right] = 0
\]

17.

\[
\begin{align*}
&\sum [a_4, a_5, a_6] \cdot [a_1, a_2, a_3] \\
&\sum [a_7, a_8, a_9] \cdot [a_4, a_5, a_6] \\
&\sum [a_0, a_1, a_2] \cdot [a_7, a_8, a_9] \\
&\sum \vdots 
\end{align*}
\]

Introducing the values of $R(a_1)$, $R(a_2)$, etc., it is not difficult to see that the first line of this equation may be written into the following form,

\[
\frac{1}{[a_1, a_2, a_3]} \left[ \begin{array}{c} [a_4, a_5, a_6] \cdot [a_1, a_2, a_3] \\ [a_7, a_8, a_9] \cdot [a_4, a_5, a_6] \\ [a_0, a_1, a_2] \cdot [a_7, a_8, a_9] \\ \vdots \end{array} \right] \right] = 0
\]

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are obtained by changing $l$ into $m$, $n$ and $p$ respectively. We have then

\[
\mathbf{A} = \mathbf{B} = \mathbf{C} = \mathbf{D} = [a_0, a_1, a_2] = \text{say } \Delta.
\]

Write for convenience

\[
[a_0, a_1, a_2] \cdot [a_1, a_2, a_3] = [a_0, a_1, a_2] \cdot P_0 = \Gamma,
\]

\[
[a_4, a_5, a_6] \cdot [a_1, a_2, a_3] = [a_4, a_5, a_6] \cdot P_1 = \Gamma_1,
\]

\[
[a_7, a_8, a_9] \cdot [a_4, a_5, a_6] = [a_7, a_8, a_9] \cdot P_2 = \Gamma_2,
\]

\[
[a_0, a_1, a_2] \cdot [a_7, a_8, a_9] = [a_0, a_1, a_2] \cdot P_3 = \Gamma_3.
\]

The term in the $\{\}$ is easily seen to be equal to $[a_0, a_1, a_2, a_3]$. Equation 17 thus takes the form,

\[
[a_4, a_5, a_6] \cdot P_1 = [a_0, a_1, a_2] \cdot P_3 + [a_4, a_5, a_6] \cdot P_2 = [a_4, a_5, a_6] \cdot P_3 + [a_0, a_1, a_2, a_3] \\
[a_4, a_5, a_6] \cdot P_2 = [a_0, a_1, a_2] \cdot P_3 + [a_4, a_5, a_6] \cdot P_1
\]

18.

Remembering to pay attention to the signs, we may write the term in $\{\}$ as $\Sigma$, then writing

\[ \Sigma = \lambda, \quad \Sigma = \mu, \quad \Sigma = \nu, \quad \Sigma = \pi, \]

we have

\[ \Sigma = \lambda, \quad \Sigma = \mu, \quad \Sigma = \nu, \quad \Sigma = \pi, \]

19.

\[ \Sigma = \lambda, \quad \Sigma = \mu, \quad \Sigma = \nu, \quad \Sigma = \pi, \]

20.

\[ \Sigma = \lambda, \quad \Sigma = \mu, \quad \Sigma = \nu, \quad \Sigma = \pi, \]

Now writing as above

\[ \theta = -[a_0, a_1, a_2, a_3] \]

and introducing the abbreviations $a$, $b$, $c$, $d$, and $\lambda$, $\mu$, $\nu$, $\pi$, it is not difficult to see that

\[ \theta = [a_0, a_1, a_2, a_3] \]

and referring now to equation 9 we have

\[ \theta = [a_0, a_1, a_2, a_3] \]

Expanding equation 18 it becomes

\[ \lambda \cdot [a_0, a_1, a_2, a_3] \cdot P_0 = [a_4, a_5, a_6, a_7, a_8, a_9] \]

\[ + [a_0, a_1, a_2, a_3] \cdot [a_4, a_5, a_6, a_7, a_8, a_9] \]

\[ + [a_0, a_1, a_2, a_3] \cdot [a_4, a_5, a_6, a_7, a_8, a_9] \]

Of course in all these summations particular care must be taken to give the right signs to each term; for example, $\Sigma = [a_0, a_1, a_2, a_3]$ means

\[ [a_0, a_1, a_2, a_3] \]

Using now equations 19 to 21 inclusive, we have, after simple reductions, for the reduced form of equation 18,

\[ [a_0, a_1, a_2, a_3] \cdot P_0 = [a_0, a_1, a_2, a_3] \cdot P_0 + [a_4, a_5, a_6, a_7, a_8, a_9] \]

\[ [a_0, a_1, a_2, a_3] \cdot P_0 = [a_0, a_1, a_2, a_3] \cdot P_0 + [a_4, a_5, a_6, a_7, a_8, a_9] \]

\[ [a_0, a_1, a_2, a_3] \cdot P_0 = [a_0, a_1, a_2, a_3] \cdot P_0 + [a_4, a_5, a_6, a_7, a_8, a_9] \]

\[ [a_0, a_1, a_2, a_3] \cdot P_0 = [a_0, a_1, a_2, a_3] \cdot P_0 + [a_4, a_5, a_6, a_7, a_8, a_9] \]
and by making the above substitution and eliminating \( P_1 \) between the two
equations thus formed we would again arrive at (6). It is then clear that the
functions with the indices
\[0, k, kl, km, kn, kp, kq, kr, ks, kt,\]
form a group of ten, such that any five being selected the squares of any of
the remaining five can be expressed as a linear function of the squares of the chosen
five. There are of course in all nine such groups, and these may be tabulated
as follows:
\[
\begin{align*}
0 & \quad kl & \quad km & \quad kp & \quad kq & \quad kr & \quad ks & \quad kt \\
0 & \quad l & \quad m & \quad n & \quad p & \quad q & \quad r & \quad s & \quad t
\end{align*}
\]

We will now take up the case of three indices, which as will be seen,
divides into two sub-cases, according to the choice of the index. The two
sub-cases give rise to two tables, the first containing 36 groups and the second
containing 84 groups. As the method of working out these groups by Brunell's
method has already been sufficiently indicated, I shall, in what follows, leave
out as much as possible the purely algebraical processes of reduction, as
they have become very long and wholly uninteresting. Squaring the function \( P_{36} \)
we have
\[ 27. \quad P_{36} = \frac{1}{\vert \begin{vmatrix} x_{1} & x_{2} & x_{3} \\ x_{2} & x_{3} & x_{1} \\ x_{3} & x_{1} & x_{2} \end{vmatrix} \vert} \left[ x_{1} x_{2} x_{3} R(x_{1}) x_{1} x_{2} x_{3} x_{1} x_{2} x_{3} x_{1} x_{2} x_{3} \right] + \ldots \]

The radicals \( R(x_{1}) R(x_{2}) \), etc., may be eliminated between any five equations of the
form 27, or between five equations of this form having each a common
index, say \( k \), or between four equations having each two indices, say \( k \) and \( l \),
common. Choose multipliers \( A, B, C \) and \( D \), such that the coefficient of

\[ R(x_{1}) R(x_{2}) \]

and in consequence the coefficients of all the other radicals shall be zero in the
sum \( A P_{36} + B P_{36} + C P_{36} + D P_{36} \).

This coefficient is easily seen to be
\[
A \vert \begin{vmatrix} x_{1} & x_{2} & x_{3} \\ x_{2} & x_{3} & x_{1} \\ x_{3} & x_{1} & x_{2} \end{vmatrix} \vert + B \vert \begin{vmatrix} x_{1} & x_{2} & x_{3} \\ x_{2} & x_{3} & x_{1} \\ x_{3} & x_{1} & x_{2} \end{vmatrix} \vert + C \vert \begin{vmatrix} x_{1} & x_{2} & x_{3} \\ x_{2} & x_{3} & x_{1} \\ x_{3} & x_{1} & x_{2} \end{vmatrix} \vert + D \vert \begin{vmatrix} x_{1} & x_{2} & x_{3} \\ x_{2} & x_{3} & x_{1} \\ x_{3} & x_{1} & x_{2} \end{vmatrix} \vert
\]
Striking out the common factor \( |a_0, a_2| |a_1, a_3| |a_4, a_5| \)
the condition to be satisfied is

\[ A |a, a_1| |a_2, a_3| + B |a, a_2| |a_1, a_3| + C |a, a_3| |a_1, a_2| + D = 0; \]

this is equivalent to

\[ a_0 A + a_1 B + a_2 C = 0, \]

\[ a_0 A + a_1 B + a_2 C + D = 0. \]

These are easily seen to be satisfied by the values

\[ A = |a, a_1|, \ B = |a, a_2|, \ C = |a, a_3|, \ D = -|a, a_1, a_2|. \]

Introducing then these values of \( A, B, C \) and \( D \), we have

\[ \frac{1}{|a_0, a_2|} \left( |a_0, a_2| R(a_1) \left[ \frac{|a_0, a_2|}{|a_0, a_2|} |a_0, a_2| \right] + \ldots \right) + \frac{1}{|a_0, a_2|} \left( |a_0, a_2| R(a_2) \left[ \frac{|a_0, a_2|}{|a_0, a_2|} |a_0, a_2| \right] + \ldots \right) + \ldots \]

\[ + \frac{1}{|a_0, a_2|} \left( |a_0, a_2| R(a_3) \left[ \frac{|a_0, a_2|}{|a_0, a_2|} |a_0, a_2| \right] + \ldots \right) \]

This is to be reduced just as in the case of two indices, viz. expand 30, so that the first line becomes

\[ |a_0, a_2| |a_0, a_2| |a_0, a_2| |a_0, a_2| |a_0, a_2| |a_0, a_2| |a_0, a_2| |a_0, a_2| |a_0, a_2| |a_0, a_2| + \ldots \]

There are three more terms similar to this to be obtained by simply advancing certain of the subscripts. The remaining three lines in 30 are to be expanded in a similar manner, and then the terms which have been introduced will disappear by aid of equations 28 and 29. The right-hand side of 30 is now easily reduced by aid of the following identities:

\[ |a_0, a_2| - |a_0, a_2| + |a_0, a_2| - |a_0, a_2| = 0, \]

\[ (a_0 + a_1 + a_2) |a_0, a_2| - (a_0 + a_1 + a_2) |a_0, a_2| = 0, \]

\[ (a_0 + a_1 + a_2) |a_0, a_2| - (a_0 + a_1 + a_2) |a_0, a_2| = 0, \]

\[ (a_0 + a_1 + a_2) |a_0, a_2| - (a_0 + a_1 + a_2) |a_0, a_2| - (a_0 + a_1 + a_2) |a_0, a_2| = 0, \]

\[ (a_0 + a_1 + a_2) |a_0, a_2| - (a_0 + a_1 + a_2) |a_0, a_2| = 0. \]
A similar group of identities may easily be written down for the general case. Using these last identities and taking the four functions

\[ P_{1m}, P_{2n}, P_{1n}, P_{2m} \]

we can, by multiplying the first by \( A \), the second by \( B \), etc., and taking the sum, eliminate the radicals \( \sqrt{R(z_1)R(z_2)}, \sqrt{R(z_1)}R(z_2) \), etc. Forming this sum it is only necessary to show that

\[ A = \{a, a, a, a\}, B = \{-a, a, a, a\}, C = \{-a, a, a, a\}, D = \{-a, a, a, a\}. \]

We have now

\[
\begin{align*}
\{a, a, a, a\} P_{1m} - \{a, a, a, a\} P_{2n} + \{a, a, a, a\} P_{1n} - \{a, a, a, a\} P_{2m} = \\
\{a, a, a, a\} \left[ \{a, a, a, a\} R(z_1) \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} + \ldots \right].
\end{align*}
\]

This may be briefly written in the form

\[
\begin{align*}
\{a, a, a, a\} P_{1m} - \{a, a, a, a\} P_{2n} + \{a, a, a, a\} P_{1n} - \{a, a, a, a\} P_{2m} = \\
\{a, a, a, a\} \left[ \{a, a, a, a\} R(z_1) \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} + \ldots \right].
\end{align*}
\]

Expanding this just as in the case of two indices and the case of equation 30, we have for the first line on the right-hand side of the equation

\[
\begin{align*}
\{a, a, a, a\} \{a, a, a, a\} & \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \\
\times & \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\}
\end{align*}
\]

+ three similar terms.

The remaining three lines of 41 are to be expanded in the same manner, and then it will at once be seen that the extra terms which have been introduced will vanish on account of the relations 29 and 30. Consider now the terms containing the factor \( R(z_2) \); they have obviously the common factor

\[
\{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\}
\]

and the remaining factor is

\[
\{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\}.
\]

Expanding \( \Sigma \) and using the identities 32 and 33, we have

\[
K = \{-a, a, a, a\}.
\]

The first line on the right-hand side of equations 40 or 41 contains four terms, the first of which contains the factor already mentioned, viz.

\[
\{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\}.
\]

the factor in each of the remaining terms is derived from this by changing \( a \) into \( n, p, q \) respectively. The same is true for the remaining three lines on the right-hand side of 40 or 41. This factor is independent of \( a \), and the others not written down are equally independent of \( n, p, \) or \( q \); for writing 43 put in

\[
\begin{align*}
\{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\}
\end{align*}
\]

and this is equal to

\[
\begin{align*}
\{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\}
\end{align*}
\]

The first three terms of this vanish by virtue of the identities 32, and the fourth term by 33 becomes

\[
\{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\}
\]

The right-hand side of 40 and 41 thus contains the factor

\[
\{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\}
\]

The right-hand member of 41 takes now the form

\[
\{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\}
\]

The \( \Sigma \) of course refers only to the cyclic permutations of the suffixes 1, 2, 3, 4.

We have now to determine the value of the quantity under the summation sign, viz.

\[
\begin{align*}
\Sigma \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\} \{a, a, a, a\}
\end{align*}
\]

in order to find the relation connecting the squares of \( P_{1m}, P_{2n}, P_{1n}, P_{2m} \).

For greater generality and completeness, however, it is better to go back to equation 30 and reduce it, i.e., find the linear relation connecting the squares of \( P_{1m}, P_{2n}, P_{1n}, P_{2m} \).

It will then be seen that by making the substitution

\[
\begin{align*}
& k l m n p q r s t \\
& l m n p q r s t
\end{align*}
\]
and eliminating one quantity, we arrive at what we might obtain directly by completing the reduction of equation 40.

The second factor in 31 is easily seen to be

\[ \frac{1}{4} \alpha, \alpha, \beta, \beta, \gamma, \gamma \]

and the same is true for the corresponding factors in the remaining three lines of 30. Now adding together the first factors of the four lines in 30, viz. those similar to the first factor in 31, we have

\[
\begin{align*}
|a, a, b, b, c, c| \sum_{x, y, z, w} & \left[ x, y, z, w \right] \\
+ |a, a, b, b, c, c| \sum_{x, y, z, w} & \left[ y, z, w \right] \\
+ |a, a, b, b, c, c| \sum_{x, y, z, w} & \left[ z, w \right] \\
+ |a, a, b, b, c, c| \sum_{x, y, z, w} & \left[ w \right]
\end{align*}
\]

The fourth line need not be written down, as its second factor is zero.

Adding these terms we have

\[
\sum_{x, y, z, w} |a, a, b, b, c, c| \left[ x, y, z, w \right] + |a, a, b, b, c, c| \left[ y, z, w \right] + |a, a, b, b, c, c| \left[ z, w \right] + |a, a, b, b, c, c| \left[ w \right]
\]

Now

\[
|a, a, b, b, c, c| + |a, a, b, b, c, c| + |a, a, b, b, c, c| + |a, a, b, b, c, c| = 0,
\]

and

\[
|a, a, b, b, c, c| + |a, a, b, b, c, c| + |a, a, b, b, c, c| + |a, a, b, b, c, c| = 0
\]

so that the above reduces to

\[
|a, a, b, b, c, c| \sum_{x, y, z, w} \left[ x, y, z, w \right] + |a, a, b, b, c, c| \sum_{x, y, z, w} \left[ y, z, w \right] + |a, a, b, b, c, c| \sum_{x, y, z, w} \left[ z, w \right] + |a, a, b, b, c, c| \sum_{x, y, z, w} \left[ w \right]
\]

the summation referring to the subscripts 1, 2, 3, 4. Equation 30, or equation 31, becomes now, by taking into account 44 and 45,

\[
\sum_{x, y, z, w} |a, a, b, b, c, c| \sum_{x, y, z, w} \left[ x, y, z, w \right] + |a, a, b, b, c, c| \sum_{x, y, z, w} \left[ y, z, w \right] + |a, a, b, b, c, c| \sum_{x, y, z, w} \left[ z, w \right] + |a, a, b, b, c, c| \sum_{x, y, z, w} \left[ w \right]
\]

For brevity write as before

\[
\sum_{x, y, z, w} = \lambda, \quad \sum_{x, y, z, w} = \mu, \quad \sum_{x, y, z, w} = \nu, \quad \sum_{x, y, z, w} = \pi
\]

the summations extending over the subscripts 1, 2, 3, 4. We have now

\[
\sum_{x, y, z, w} |a, a, b, b, c, c| \sum_{x, y, z, w} \left[ x, y, z, w \right] + |a, a, b, b, c, c| \sum_{x, y, z, w} \left[ y, z, w \right] + |a, a, b, b, c, c| \sum_{x, y, z, w} \left[ z, w \right] + |a, a, b, b, c, c| \sum_{x, y, z, w} \left[ w \right]
\]

\[
\sum_{x, y, z, w} (x - y + z + w) + (x - y + z + w) + (x - y + z + w) + (x - y + z + w)
\]

\[
=x(x - y + z + w) + (x - y + z + w) + (x - y + z + w) + (x - y + z + w)
\]

\[
= 4(x - y + z + w)
\]

47. 

\[
\sum_{x, y, z, w} (x - y + z + w) + (x - y + z + w) + (x - y + z + w) + (x - y + z + w)
\]

\[
= 4(x - y + z + w)
\]

\[
= 4(x + y - z - w)
\]

\[
= 4(x + y + z + w)
\]

\[
= 4(x + y + z + w)
\]

\[
= 4(x + y + z + w)
\]

\[
= 4(x + y + z + w)
\]

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= 4(x + y + z + w)
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= 4(x + y + z + w)
\]

\[
= 4(x + y + z + w)
\]

\[
= 4(x + y + z + w)
\]
Craig: On Quadruple Theta-Functions.

If instead of eliminating $P_q$ between $48$ and $49$ we had eliminated $P_1$, we would arrive at a linear relation connecting

$$P_{nm}, P_{nt}, P_{tp}, P_{tn}, P_{tt}, P_{pt}, P_{tp}.$$  

say

51. $$A^tP_{nm} + B^tP_{nt} + C^tP_{tp} + D^tP_{tn} + E^tP_{tt} + F^tP_{pt} + G^tP_{tp} = 0.$$  

Now effect upon $50$ the substitution

$$m, n, p, q, r, t, s, m$$  

say $\Omega$,

then we will obviously obtain a relation of the form,

52. $$A'^tP_{nm} + B'^tP_{nt} + C'^tP_{tp} + D'^tP_{tn} + E'^tP_{tt} + F'^tP_{pt} + G'^tP_{tp} = 0.$$  

Eliminating $P_1$ between $51$ and $52$ and we have a linear relation connecting five of the $P$-functions possessing triple indices, and one possessing a single index, viz., a relation of the form

53. $$A^xP_{nm} + B^xP_{nt} + C^xP_{tp} + D^xP_{tn} + E^xP_{tt} + F^xP_{pt} = 0.$$  

Or, if $P_1$ had been eliminated, we would have a relation connecting five $P$-functions with triple indices, and one with a double index, viz.

54. $$A^yP_{nm} + B^yP_{nt} + C^yP_{tp} + D^yP_{tn} + E^yP_{tt} + F^yP_{pt} + G^yP_{tp} = 0.$$  

Effecting the substitution $\Omega$ upon $54$ and there results an equation of the form,

55. $$A'^yP_{nm} + B'^yP_{nt} + C'^yP_{tp} + D'^yP_{tn} + E'^yP_{tt} + F'^yP_{pt} + G'^yP_{tp} = 0.$$  

Now finally eliminating $P_1$ between $54$ and $55$ and we arrive at a linear relation connecting the squares of six $P$-functions with triple indices, viz.

56. $$A^xP_{nm} + B^xP_{nt} + C^xP_{tp} + D^xP_{tn} + E^xP_{tt} + F^xP_{pt} = 0.$$  

Of course, the substitution $\Omega$ performed upon $56$ would give

57. $$A'^yP_{nm} + B'^yP_{nt} + C'^yP_{tp} + D'^yP_{tn} + E'^yP_{tt} + F'^yP_{pt} + G'^yP_{tp} = 0.$$  

We arrive thus at the conclusion that the ten functions

$$P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}$$

form a group such that selecting any five of them the square of any one of the remaining five is linearly expressible in terms of the squares of the chosen five.

There are in all $36$ such groups, and they are given in the following table:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$l$</th>
<th>$kl$</th>
<th>$km$</th>
<th>$kn$</th>
<th>$knp$</th>
<th>$kmp$</th>
<th>$kp$</th>
<th>$kq$</th>
<th>$kr$</th>
<th>$ks$</th>
<th>$kt$</th>
</tr>
</thead>
<tbody>
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<td>$knp$</td>
<td>$kmp$</td>
<td>$kmr$</td>
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<tr>
<td>$n$</td>
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<td>$kmn$</td>
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<tr>
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<td>$kql$</td>
<td>$kqm$</td>
<td>$kmr$</td>
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</tr>
<tr>
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<td>$ks$</td>
<td>$kls$</td>
<td>$kmr$</td>
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</tbody>
</table>
Taking out the common factor
\[ A \left[ a_1, a_2, a_3, a_4 \right][a_5, a_6, a_7, a_8][a_9], \]
and this becomes
\[ A [a_1, a_2, a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3], \]
\[ + B [a_1, a_2, a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3], \]
\[ + C [a_1, a_2, a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3], \]
\[ - D [a_1, a_2, a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3], \]
\[ = 0. \]

Introducing here the values
\[ A [a_1, a_2, a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3], \]
\[ B [a_1, a_2, a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3], \]
\[ C [a_1, a_2, a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3], \]
\[ D [a_1, a_2, a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3], \]
we ought to be able to show that the squares of any six of the functions whose
indices are
\[ \ld \ \text{all except one}, \]
are connected by a linear relation. We would then have a table of 84 groups
similar to the above, and such that the squares of any six functions in a given
group are connected by a linear relation. In the case of quadruple indices there
would also be a second table containing 126 groups, of which

\[ \text{we have to be able to show that the squares of any six of the functions whose}
\text{indices are} \]

\[ \ld \ \text{all except one}, \]

\[ \text{are connected by a linear relation. We would then have a table of 84 groups}
\text{similar to the above, and such that the squares of any six functions in a given}
\text{group are connected by a linear relation. In the case of quadruple indices there}
\text{would also be a second table containing 126 groups, of which}
\]

\[ \text{we have to be able to show that the squares of any six of the functions whose}
\text{indices are} \]

\[ \ld \ \text{all except one}, \]

\[ \text{are connected by a linear relation. We would then have a table of 84 groups}
\text{similar to the above, and such that the squares of any six functions in a given}
\text{group are connected by a linear relation. In the case of quadruple indices there}
\text{would also be a second table containing 126 groups, of which}
\]

Consider now the functions with four indices and find values of \( A, B, C, D \),
such that the radicals \( \sqrt{K(x)}(x) \), etc. shall vanish in the sum
\[ A B - B C + C D - D A = 0. \]
The coefficient of \( \sqrt{K(x)}(x) \) in this sum is, leaving out the common factor
\[ [a_1, a_2][a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3], \]
\[ A [a_1, a_2][a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3] - B [a_1, a_2, a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3], \]
\[ + C [a_1, a_2, a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3] - D [a_1, a_2, a_3][a_4, a_5][a_6, a_7, a_8, a_9][a_2][a_3], \]
\[ = 0. \]
In order to show the linear relations between the squares of the $F$-functions with quadruple indices, we will begin with the 44 groups, of which

$k_{44}$ is the first. Take the functions

\[ F_{44}, F_{444}, F_{4444}, \]

and find the values of $A, B, C, D, S$ so that the radicals $\sqrt{B(a)}R(c_2)$, etc., shall vanish in the sum

\[ AP_{44} + BP_{444} + CP_{4444} + DP_{44444}. \]

Dropping out a common factor

\[ [a_4][c_4][c_4][c_4] \]

the necessary condition is easily seen to be

\[ A + B[a_4][c_4] + C[c_4][c_4] + D[c_4][c_4] = 0; \]

this is equivalent to

\[ A + B + C + D = 0, \]

\[ A + a_4B + a_4C + a_4D = 0, \]

\[ A + a_4B + a_4C + a_4D = 0, \]

and these are easily seen to be satisfied by the values

\[ A = -|a_4a_4a_4|, B = |a_4a_4a_4|, C = |a_4a_4a_4|, D = |a_4a_4a_4|. \]

Introducing these values into the above sum we have

\[ \frac{1}{|a_4a_4a_4a_4|} \]

\[ \frac{1}{[a_4a_4a_4a_4]} \]

\[ \frac{1}{[a_4a_4a_4a_4]} \]

\[ \frac{1}{[a_4a_4a_4a_4]} \]

\[ \frac{1}{[a_4a_4a_4a_4]} \]

The right-hand side of this equation may be written in the form

\[ \frac{1}{|a_4a_4a_4a_4|} \]

\[ \frac{1}{[a_4a_4a_4a_4]} \]

\[ \frac{1}{[a_4a_4a_4a_4]} \]

\[ \frac{1}{[a_4a_4a_4a_4]} \]

\[ \frac{1}{[a_4a_4a_4a_4]} \]

The omitted terms are easily supplied by symmetry. Now

\[ \Sigma |a_4a_4a_4a_4| = 0, \Sigma |a_4a_4a_4a_4| + |a_4a_4a_4a_4| = |a_4a_4a_4a_4| \]

The fourth line of the last equation vanishes and the first terms of the first three

\[ |a_4a_4a_4a_4| \]

\[ |a_4a_4a_4a_4| \]

\[ |a_4a_4a_4a_4| \]